

# The hybrid inflation waterfall and the primordial curvature perturbation<sup>\*</sup>

---

**David H. Lyth, Consortium for Fundamental Physics, Cosmology and  
Astroparticle Group, Department of Physics, Lancaster University, Lancaster  
LA1 4YB, UK**

**ABSTRACT:** Without demanding a specific form for the inflaton potential, we obtain an estimate of the contribution to the curvature perturbation generated during the linear era of the hybrid inflation waterfall. The spectrum of this contribution peaks at some wavenumber  $k = k_*$ , and goes like  $k^3$  for  $k \ll k_*$ , making it typically negligible on cosmological scales. The scale  $k_*$  can be outside the horizon at the end of inflation, in which case  $\zeta = -(g^2 - \langle g^2 \rangle)$  with  $g$  gaussian. Taking this into account, the cosmological bound on the abundance of black holes is likely to be satisfied if the curvaton mass  $m$  much bigger than the Hubble parameter  $H$ , but is likely to be violated if  $m \lesssim H$ . Coming to the contribution to  $\zeta$  from the rest of the waterfall, we are led to consider the use of the ‘end-of-inflation’ formula, giving the contribution to  $\zeta$  generated during a sufficiently sharp transition from nearly-exponential inflation to non-inflation, and we state for the first time the criterion for the transition to be sufficiently sharp. Our formulas are applied to supersymmetric GUT inflation and to supernatural/running-mass inflation.

**KEYWORDS:** Primordial curvature perturbation.

---

<sup>\*</sup>A preliminary version of this paper appeared as arXiv:1107.1681

---

## Contents

<b>1. Hybrid inflation</b>	<b>2</b>
1.1 Scales leaving the horizon	2
1.2 Hybrid inflation potential	2
1.3 Standard Scenario	4
1.4 The waterfall	5
<b>2. Waterfall field during the linear era</b>	<b>6</b>
2.1 Evolution of $\chi$	6
2.2 Classical field $\chi(\mathbf{x}, t)$	8
2.3 End of the linear era	9
2.4 Energy density and pressure of $\chi$	10
2.5 Justifying the neglect of $\overline{\chi}$	11
<b>3. Contribution <math>\zeta_\chi</math> to the curvature perturbation</b>	<b>12</b>
3.1 The contribution $\zeta_{\text{lin}}$	13
3.2 The contribution $\zeta_{\text{nl}}$	14
3.3 Other uses of the ‘end of inflation’ formula	16
<b>4. Effect of <math>\zeta_\chi</math></b>	<b>16</b>
4.1 Cosmological black hole bound on $\mathcal{P}_\zeta$	16
4.2 The effect of $\zeta_{\text{lin}}$	18
<b>5. Estimates using a simple approximation</b>	<b>19</b>
5.1 The approximation for $m^2(t)$	19
5.2 Slow-roll approximation	20
5.3 Trading $\mu$ for $f$	21
5.4 The case $t_{\text{end}} < t_*$	22
5.5 The case $t_{\text{end}} > t_*$	23
5.6 Duration of the non-linear era	24
5.7 Two inflation models	24
5.7.1 Supersymmetric GUT hybrid inflation	25
5.7.2 Supernatural/running-mass inflation	25
<b>6. The case <math>m \sim H</math></b>	<b>26</b>
<b>7. Comparison with other calculations</b>	<b>27</b>
<b>8. Conclusion</b>	<b>28</b>

## 1. Hybrid inflation

Hybrid inflation [1, 2, 3] ends with a phase transition known as the waterfall, which up to now has been studied only in special cases. This paper, which is a continuation of [4], provides a rather general treatment. We begin by defining the setup.

### 1.1 Scales leaving the horizon

An inflation model starts to make contact with observation only around the time that the observable universe leaves the horizon. The following description hybrid inflation is intended to apply to the subsequent era.

Within the standard cosmology, the number  $N_{\text{obs}}$  of  $e$ -folds of inflation after the observable universe leaves the horizon satisfies [5]

$$63 - \frac{1}{2} \ln \frac{10^{-5} M_{\text{P}}}{H} \lesssim N_{\text{obs}} \lesssim 49 - \frac{1}{3} \ln \frac{10^{-5} M_{\text{P}}}{H}. \quad (1.1)$$

(The time-dependence of  $H$  is ignored in this expression, which is usually a good approximation.) The upper bound corresponds to matter domination from the end of inflation to the epoch  $T = 1 \text{ MeV}$ , with radiation domination thereafter until the observed matter dominated era, while the lower bound replaces the former era by one of radiation domination.

The scales probed by observation of large scale structure (cosmological scales) leave the horizon during the first 15 or so  $e$ -folds after the observable universe. On these scales, the curvature perturbation  $\zeta$  is nearly gaussian with a nearly scale-invariant spectrum  $\mathcal{P}_{\zeta}(k) \sim (5 \times 10^{-5})^2$ .

### 1.2 Hybrid inflation potential

Our analysis applies to a wide class of hybrid inflation models. The essential features of the potential are captured by the following expression;

$$V(\phi, \chi) = V_0 + V(\phi) + \frac{1}{2} m^2(\phi) \chi^2 + \frac{1}{4} \lambda \chi^4 \quad (1.2)$$

$$m^2(\phi) \equiv g^2 \phi^2 - m^2 \equiv g^2 (\phi^2 - \phi_c^2). \quad (1.3)$$

To have a perturbative quantum theory we demand  $g \ll 1$  and  $\lambda \ll 1$ . The inflaton  $\phi$  is supposed to have zero vev and  $V(\phi)$  is set to zero at the vev. We require  $V'(\phi) > 0$  during inflation so that  $\phi$  moves towards its vev. The era of inflation with  $\phi < \phi_c$  is called the waterfall.

The requirements that  $V$  and  $\partial V/\partial\chi$  vanish in the vacuum determine  $V_0$  and the vev of the waterfall field  $\chi$ :

$$\chi_{\text{vev}}^2 = m^2/\lambda, \quad V_0 = m^4/4\lambda. \quad (1.4)$$

It is necessary for our analysis that the waterfall field  $\chi$  has the canonical kinetic term. For simplicity we will pretend that  $\chi$  is a single real field. At least within the Standard Scenario defined below, that cannot really be the case because it would lead to the formation of domain walls, located along surfaces at which  $\chi(\mathbf{x}, t)$  is trapped at the origin, which would be fatal to the cosmology. In reality,  $\chi$  will be replaced by a function of two or more real fields. So that there is only one effective degree of freedom in the  $\chi$  direction, we will demand that the function is invariant under some symmetry group of the action. Then the only change in our analysis for the realistic case would be the introduction of some numerical factors into the equations. In the realistic case the domain walls might be replaced by cosmic strings or monopoles, but in general the trapping of  $\chi$  will not occur and  $\chi(\mathbf{x}, t)$  will everywhere approach its vev.

The inflaton  $\phi$  may also be replaced by a function of two or more real fields. If there is still only one effective degree of freedom the only change is again the introduction of numerical factors. In the opposite case of multi-field inflation, corresponding to a family of inflationary trajectories that are curved in field space, most of our analysis still applies if, by the onset of the waterfall, the family has collapsed to a single effective trajectory which has negligible curvature during the waterfall. To obtain powerful results we take the inflaton to have the canonical kinetic term, though much of our analysis would apply to, for instance, k-inflation [6].

Hybrid inflation was first discovered in the context of single-field inflation [1, 2]. It was given its name in [2], where the form (1.2) was invoked for  $V(\phi, \chi)$  with  $V(\phi) = m_\phi^2 \phi^2/2$ . With parameters chosen to give the Standard Scenario, and demanding also that  $\delta\phi$  is responsible for the observed curvature perturbation, this gives spectral index  $n > 1$  in contradiction with observation. Many forms of  $V(\phi)$  have been proposed, which allow  $\delta\phi$  to generate the curvature perturbation [7, 5] within the single-field inflation scenario.

In our calculations we employ Eq. (1.2) for  $V(\phi, \chi)$ , without specifying the inflaton potential  $V(\phi)$ . Minor variants of Eq. (1.2) would make little difference. The interaction  $g^2 \phi^2 \chi^2$  might be replaced by  $\phi^2 \chi^{2+n}/\Lambda^n$  where  $\Lambda$  is a uv cutoff, or the term  $\lambda \chi^4$  might be replaced by  $\chi^{4+n}/\Lambda^n$ . For our purpose, these variants are equivalent to allowing (respectively)  $g$  and  $\lambda$  to be many orders of magnitude below unity.

More drastic modifications of Eq. (1.2) have been proposed, including inverted hybrid inflation [8] where  $\phi$  is increasing during inflation, as well as mutated and smooth hybrid inflation [9, 10] where the waterfall field varies during inflation. Also, the waterfall potential might have a local minimum at the origin so that the waterfall proceeds by bubble formation [11, 3]. Our analysis does not apply to those cases.

### 1.3 Standard Scenario

By varying the parameters in the potential (1.2), one can have a wide range of scenarios that is still not fully explored. Most discussions of hybrid inflation make some assumptions, corresponding to what might be called the Standard Scenario. In this section we state those assumptions, which are made in the rest of the paper.

Until  $\chi$  approaches its vev at the end of the waterfall, inflation is supposed to be nearly exponential ( $\epsilon_H \equiv |\dot{H}|/H^2 \ll 1$ ) with  $V_0$  dominating the potential:

$$3M_{\text{P}}^2 H^2(t) = \rho(t) \simeq V_0. \quad (1.5)$$

We take  $H$  to be constant during the waterfall, which is usually a good approximation. Nearly exponential inflation requires

$$\dot{\phi}^2 \ll 3M_{\text{P}}^2 H^2. \quad (1.6)$$

Eqs. (1.4) and (1.5) give

$$\chi_{\text{vev}}^2/M_{\text{P}}^2 \simeq 12H^2/m^2. \quad (1.7)$$

It is usually supposed that  $\chi_{\text{vev}} \ll M_{\text{P}}$  corresponding to  $m \gg H$ . (In particular, GUT inflation [12, 13] takes  $\chi$  to be a GUT Higgs field with  $\chi_{\text{vev}} \sim 10^{-2}M_{\text{P}}$ .) One sometimes considers  $\chi_{\text{vev}}$  roughly of order  $M_{\text{P}}$  corresponding to  $m$  roughly of order  $H$  (supernatural [14] and running mass [15] inflation). There do not seem to be any papers considering  $\chi_{\text{vev}} \gg M_{\text{P}}$  which would correspond to  $m \ll H$ .

Using Eqs. (1.4) and (1.5), the requirement  $\lambda \ll 1$  is equivalent to

$$m/H < \sqrt{M_{\text{P}}/H} \quad (\lambda \ll 1). \quad (1.8)$$

Successful BBN and the upper bound on the tensor perturbation require [5]

$$10^{-42} < H/M_{\text{P}} < 10^{-5} \quad (\text{BBN and tensor}). \quad (1.9)$$

The upper part of the range is favoured, especially [16] because we deal with hybrid inflation.

One usually requires  $\phi \ll M_{\text{P}}$  but we will just invoke the weaker requirement<sup>#1</sup>

$$\phi_c \equiv m/g \ll M_{\text{P}}. \quad (1.10)$$

If  $\phi$  is big enough we have  $m^2(\phi) \gtrsim H^2$ . Then we assume that  $\chi$  vanishes up to a vacuum fluctuation which is set to zero. If  $\phi$  is small enough,  $m^2(\phi) \lesssim -H^2$ . Hence there is a ‘transition’ regime with  $|m^2(\phi)| \ll H^2$ . If the transition takes several Hubble times, the quantum fluctuation of  $\chi$  will be converted to a classical perturbation, with spectrum  $\sim (H/2\pi)^2$  on all scales leaving the horizon during the

---

<sup>#1</sup>This is also invoked in our earlier paper [4] but note that Eq. (80) of [4] has a typo.

transition. To avoid this the transition should take less than a Hubble time or so (fast transition).

The waterfall starts at  $m^2(\phi) = 0$  which is in the middle of the transition. During the waterfall the vacuum fluctuation of  $\chi$  is converted to a classical field  $\chi(\mathbf{x}, t)$ , with  $\chi^2$  moving towards  $\chi_{\text{vev}}^2$ . The waterfall ends when  $\chi^2(\mathbf{x}, t) \simeq \chi_{\text{vev}}^2$ , and inflation is supposed to end then because  $V(\phi)$  is not supposed to support inflation without the additional term  $V_0$ .

Regarding  $\phi$ , we require that it decreases monotonically before the waterfall, and afterward for as long as it affects the evolution of  $\chi$ . This assumption is not at all trivial, because  $V(\phi)$  may steepen as  $\phi$  decreases, causing  $\phi$  to oscillate about the origin. The evolution of  $\chi$  has yet to be studied for that case, which occurs in part of the parameter space for some well-motivated forms of the potential, including GUT inflation [12, 13] and running-mass inflation [15].

#### 1.4 The waterfall

During the waterfall we need to consider both  $\phi$  and  $\chi$ . Taking both fields to live in unperturbed spacetime (ie. ignoring back-reaction) the evolution equations during the waterfall are

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) - \nabla^2\phi = -g^2\chi^2\phi \quad (1.11)$$

$$\ddot{\chi} + 3H\dot{\chi} + m^2(\phi)\chi - \nabla^2\chi = -\lambda\chi^3. \quad (1.12)$$

We assume that the waterfall starts with an era during which Eq. (1.12) can be replaced by<sup>#2</sup>

$$\ddot{\chi}_{\mathbf{k}} + 3H\dot{\chi}_{\mathbf{k}} + [(k/a)^2 + m^2(t)]\chi_{\mathbf{k}} = 0, \quad (1.13)$$

with  $m^2(t)$  is independent of  $\chi$ . We call this the linear era, and it will be our main focus. Regarding  $\chi_{\mathbf{k}}(t)$  as an operator [4, 5], its mode function  $\chi_k(t)$  also satisfies Eq. (1.13). We will see how  $\chi_k(t)$  grows exponentially for suitably small  $k$ , generating a classical quantity  $\chi_{\mathbf{k}}(t)$ . Keeping only the classical modes, we arrive at a classical field  $\chi$ .

During at least the first part of the linear era,  $m^2(\phi)$  depends significantly on  $\phi$ . Then the right hand side of Eq. (1.11) has to be negligible so that  $m^2(t)$  can be independent of  $\chi$ . With that condition in place we just have to worry about the perturbation  $\delta\phi$  that is generated from the vacuum fluctuation.

If the linear era of the waterfall takes no more than a Hubble time or so,  $\delta\phi$  can be completely eliminated by taking the spacetime slicing to be one of uniform  $\phi$ . But if the waterfall takes much more than a Hubble time, new contributions to  $\delta\phi$  are generated as each scale leaves the horizon. To avoid this quantum effect on

---

<sup>#2</sup>This assumption implies some lower bound on  $|\dot{\phi}|$  but it is not clear how to calculate the bound [4, 17].

the evolution  $\chi$ , we have to assume that the new contributions to  $\delta\phi$  are negligible. Then we can again choose the slicing of uniform  $\phi$ .<sup>#3</sup>

For the threading of spacetime, we choose the comoving worldlines (those moving with the fluid, so that a fluid element has zero momentum density). The gradients of both  $\phi$  and (as we shall see) the classical field  $\chi$  are small compared with their time derivatives. If they vanished, the comoving worldlines would be free-falling and orthogonal to the slicing, and we could choose the time coordinate labeling the slicing to be proper time along each thread. We assume that the gradients are small enough to make that choice possible to an acceptable approximation. This completes the definition of the gauge in which the classical field  $\chi(\mathbf{x}, t)$  is defined.

We also need to justify the use of Eq. (1.13) for the mode function  $\chi_k$ , before the classical quantity  $\chi_{\mathbf{k}}(t)$  is generated. As we will see, Eq. (1.13) is needed for that purpose only for modes that are well inside the horizon during this time and (at least with the approximation of Section 5) only for much less than a Hubble time. That being the case, we can ignore the second term of Eq. (1.13) and set  $a$  equal to a constant so that Eq. (1.13) becomes a flat spacetime equation in which back-reaction is negligible.

## 2. Waterfall field during the linear era

### 2.1 Evolution of $\chi$

With  $H$  constant, we can use conformal time  $\eta = -1/aH$  to write (1.13) as

$$\frac{d^2(a\chi_{\mathbf{k}})}{d\eta^2} + \omega_k^2 a\chi_{\mathbf{k}} = 0, \quad (2.1)$$

with

$$\omega_k^2(\eta) \equiv k^2 + a^2 \tilde{m}^2(t), \quad \tilde{m}^2 \equiv m^2(t) - 2H^2, \quad m^2(t) \equiv g^2 \phi^2(t) - m^2. \quad (2.2)$$

For sufficiently small  $k$ , we can set  $\omega_k^2 \simeq \omega_{k=0}^2 = a^2 \tilde{m}^2$ . Then  $\omega_k^2$  switches from positive to negative before  $\phi = \phi_c$  (but much less than a Hubble time before, by virtue of our fast transition assumption). For  $k^2 > 0$  the switch is later. For the scales that we need to consider, we assume that there are eras both before and after the switch when  $\omega_k^2$  satisfies the adiabaticity condition  $d|\omega_k|/d\eta \ll |\omega_k^2|$ .

---

<sup>#3</sup>Since we neglect the new contributions to  $\delta\phi$  during the linear era (while  $m^2(\phi)$  depends significantly on  $\phi$ ), we neglect also their effect on the spectrum  $\mathcal{P}_\zeta(k)$ . That is presumably a good approximation if  $\mathcal{P}_{\zeta_\phi}$  given by Eq. (5.12) is much less than the contribution  $\mathcal{P}_{\zeta_{\text{lin}}}(k)$  that we are going to calculate. That will probably be the case if  $\mathcal{P}_{\zeta_{\text{lin}}}(k)$  is big enough to form black holes, which is our main concern. It will not be the case if we deal with one of those exceptional inflation models where  $\mathcal{P}_{\zeta_\phi}(k)$ , on the scales leaving the horizon during the linear era of the waterfall, is itself big enough to form black holes. Then we have the opposite situation: Eq. (5.12) will be valid if  $\mathcal{P}_{\zeta_{\text{lin}}}$  is *not* big enough to form black holes.

Taking  $\chi_k$  to be an operator, its mode function  $\chi_k$  satisfies Eq. (2.1). During the adiabaticity era before the switch we take the mode function to be

$$a\chi_k = (2\omega_k(\eta))^{-1/2} \exp\left(-i \int^\eta \omega_k(\eta) d\eta\right), \quad (2.3)$$

which defines the vacuum state. During the adiabaticity era after the switch

$$a\chi_k \simeq (2|\omega_k(\eta)|)^{-1/2} \exp\left(\int_{\eta_1(k)}^\eta |\omega_k(\eta)| d\eta\right), \quad (2.4)$$

where the subscript 1 denotes the beginning of the adiabatic era. The displayed prefactor is exact [4] only if  $m^2(t) \propto t$  and  $H(t - t_1) \ll 1$ . As we are about to see,  $\chi_k$  grows during this era and we call it the growth era.

During the growth era, the adiabaticity condition is equivalent to the two conditions

$$\frac{2H}{|m(t)|} \ll \left[1 - \left(\frac{k}{a(t)|m(t)|}\right)^2\right]^{3/2} \quad (2.5)$$

$$\frac{1}{|m(t)|^2} \frac{d|m(t)|}{dt} \ll \left[1 - \left(\frac{k}{a(t)|m(t)|}\right)^2\right]^{3/2}. \quad (2.6)$$

The growth era begins when both conditions are first satisfied.

The first condition implies  $|m(t)| \gg H$  so that  $|\tilde{m}(t)| \simeq |m(t)|$ , and it can hold only if  $m \gg H$ . For  $k \ll a(t)|m(t)|$  we have  $|\omega_k| \simeq |\omega_{k=0}| \simeq a(t)|m(t)|$ . With Eq. (2.4) this gives  $\dot{\chi}_k \simeq |\omega_k|\chi_k/a$ . At  $k = 0$  Eq. (2.4) becomes

$$\chi_{k=0}(t) \simeq (2a^3|m(t)|)^{-1/2} \exp\left(\int_{t_1}^t dt |m(t)|\right). \quad (2.7)$$

Ignoring the relatively slow time-dependence of the prefactor, we have as a rough approximation

$$\chi_{k=0}(t) \simeq (2a_1^3|m(t_1)|)^{-1/2} \exp\left(\int_{t_1}^t dt |m(t)|\right). \quad (2.8)$$

In the regime  $k \ll a|m(t)|$  we have

$$|\omega_k| \simeq a|m(t)| \left(1 - \frac{1}{2} \frac{k^2}{a^2|m(t)|^2}\right), \quad (2.9)$$

giving

$$\chi_k(t) \simeq \chi_{k=0}(t) e^{-k^2/2k_*^2(t)}, \quad (2.10)$$

where

$$k_*^2(t) \equiv \left(\int_{t_1}^t \frac{dt}{a^2|m(t)|}\right)^{-1}. \quad (2.11)$$



By virtue of Eq. (2.6), the change in  $|m(t)|$  in time  $|m(t)|^{-1}$  is small and so is the change in  $a$ . Defining

$$t_{\text{start}} \equiv t_1 + |m(t_1)|^{-1}, \quad (2.12)$$

we get

$$k_*(t_{\text{start}}) \simeq a(t_1)|m(t_1)|. \quad (2.13)$$

To avoid the divergence in  $k_*(t)$  at  $t = t_1$ , we will regard  $t_{\text{start}}$  as the start of the growth era rather than  $t_1$ . After  $t_{\text{start}}$ ,  $k_*(t)$  decreases while  $a|m(t)|$  increases. We assume that  $k_*^2(t) \ll a^2|m^2(t)|$ , except for a brief era near the beginning of the growth era that can be ignored.<sup>#4</sup> Then  $\chi_k(t)$  at fixed  $t$  falls exponentially in the regime  $k_*(t) \lesssim k < a|m(t)|$  and significant modes have  $k \lesssim k_*(t)$ .

The number of  $e$ -folds of growth is  $N(t) \equiv H(t - t_{\text{start}})$ . We denote the end of the linear era by a subscript ‘end’. If  $N(t_{\text{end}}) \lesssim 1$ ,  $k_*(t)$  falls continuously. If instead  $N(t_{\text{end}}) \gg 1$ , the exponential increase of  $a$  causes  $k_*(t)$  to level off after  $N(t) \sim 1$ . Using  $H \ll |m(t)| < m$ , we learn that in any case

$$1 \ll \left( \frac{k_*(t)}{a(t_{\text{start}})H} \right)^2 < \frac{m}{H}. \quad (2.14)$$

This tells us that the scale  $k_*(t)$  is shorter than the scale leaving the horizon at the beginning of the waterfall. Since we assume that cosmological scales are outside the horizon at this stage,  $k_*(t)$  is shorter than any cosmological scale. Dividing both sides by  $\exp(2N(t_{\text{end}}))$  we have

$$e^{-N(t_{\text{end}})} \ll \left( \frac{k_*(t)}{a(t_{\text{end}})H} \right) \lesssim \left( \frac{m}{H} \right)^{1/2} e^{-N(t_{\text{end}})}. \quad (2.15)$$

This tells us that scale  $k_*(t)$  (and in particular, its final value  $k_*(t_{\text{end}})$ ) can be far outside the horizon at the end of the linear era.

## 2.2 Classical field $\chi(\mathbf{x}, t)$

During the growth era the mode function  $\chi_k$  has constant phase (zero with our convention), which means that  $\chi_{\mathbf{k}}(t) \propto \chi_k(t)$  can be regarded as a classical field. The significant modes have  $k \lesssim k_*(t) \ll a(t)|m(t)|$ . Because when for each mode. This means that the continuous creation of new classical modes, occurring for each mode when  $\omega_k^2$  becomes negative at  $k \sim a(t)m(t)$ , can be ignored.

For the significant modes,  $\dot{\chi}_k/\chi_k \simeq a(t)|m(t)|$ . The classical field has approximately the same behaviour,<sup>#5</sup>

$$\dot{\chi}(\mathbf{x}, t) \simeq |m(t)|\chi(\mathbf{x}, t). \quad (2.16)$$

---

<sup>#4</sup>This assumption holds within the approximation of Section 5.

<sup>#5</sup>This behaviour breaks down near any locations with  $\chi(\mathbf{x}, t) = 0$ . To discuss them we would have to extend the discussion to a multi-component  $\chi$  as mentioned at the end of Section 1.2. We assume that if they exist, they are rare enough to be ignored.

Since  $k_*(t) \ll a|m(t)|$ , the gradient of  $\chi$  is small compared with its time derivative.

The spectrum of  $\chi$  is

$$\mathcal{P}_\chi(k, t) \equiv (k^3/2\pi^2)P_\chi(k, t) = (k^3/2\pi^2)|\chi_k(t)|^2. \quad (2.17)$$

Using Eq. (2.10) the mean square (spatial average) of  $\chi^2$  is [4]

$$\langle \chi^2(t) \rangle = \int \frac{dk}{k} \mathcal{P}_\chi(k, t) = \frac{1}{(2\pi)^3} \int d^3k P_\chi(k, t) = (2\pi)^{-3/2} P_\chi(0, t) k_*^3(t), \quad (2.18)$$

where  $P_\chi(0, t) = |\chi_{k=0}(t)|^2$  is given by Eq. (2.7).

We denote the perturbation in  $\chi^2$  by  $\delta\chi^2$ :

$$\delta\chi^2(\mathbf{x}, t) \equiv \chi^2(\mathbf{x}, t) - \langle \chi^2(t) \rangle. \quad (2.19)$$

The convolution theorem gives [18] for  $P_{\delta\chi^2} \equiv (2\pi^2/k^3)\mathcal{P}_{\delta\chi^2}$

$$P_{\delta\chi^2}(k, t) = \frac{2}{(2\pi)^3} \int d^3k' P_\chi(k', t) P_\chi(|\mathbf{k} - \mathbf{k}'|, t). \quad (2.20)$$

For  $k \ll k_*(t)$  this gives [4]

$$\mathcal{P}_{\delta\chi^2}(k, t) = \frac{1}{\sqrt{\pi}} \langle \chi^2(t) \rangle^2 [k/k_*(t)]^3. \quad (2.21)$$

For  $k \gg k_*(t)$  it gives

$$\mathcal{P}_{\delta\chi^2}(k, t) = 2\langle \chi^2(t) \rangle \mathcal{P}_\chi(k, t), \quad (2.22)$$

which falls exponentially at fixed  $t$ .

### 2.3 End of the linear era

At each location, the linear equation (2.16) ceases to be valid around the time when  $\chi^2(\mathbf{x}, t)$  achieves some value  $\chi_{\text{nl}}^2$ . This time is given by

$$\chi^2(\mathbf{x}, t_{\text{nl}}(\mathbf{x})) = \chi_{\text{nl}}^2. \quad (2.23)$$

We will take the linear epoch to end at a time  $t_{\text{end}}$ , such that the fraction  $y_>$  of space with  $\chi^2(\mathbf{x}, t) > \chi_{\text{nl}}^2$  is small. We will see that the probability distribution of  $\chi(\mathbf{x}, t)$  is gaussian, and using the approximation  $\text{erfc}(x) \sim \exp(-x^2)$  we have

$$\chi_{\text{nl}}^2 \simeq \ln(1/y_>) \langle \chi^2(t_{\text{end}}) \rangle. \quad (2.24)$$

According to this equation  $t_{\text{end}}$  is not very sensitive to the choice of  $y_>$ , and for estimates we will take  $\ln(1/y_>) \sim 1$ .

If the linear era lasts for long enough, we will have  $m^2(t_{\text{end}}) \simeq -m^2$  (ie.  $\phi(t_{\text{end}}) \ll \phi_c = m/g$ ). In that case the right hand side of Eq. (1.11) is irrelevant, and the linear era ends only when the right hand side of Eq. (1.12) becomes significant. This gives

$$\chi_{\text{nl}}^2 \simeq \chi_{\text{vev}}^2 = 12M_{\text{P}}^2 H^2 / m^2. \quad (2.25)$$

Now suppose instead that  $|m(t_{\text{end}})| \ll m$  (ie.  $\phi(t_{\text{end}}) \simeq \phi_c$ ). Then the linear era will end when the right hand side of Eq. (1.11) becomes significant, provided that the right hand side of Eq. (1.12) is then still negligible which we are about to show will be the case. If  $\phi$  is still slowly rolling so that  $3H\dot{\phi}(t) = -V'$ , the right hand side of Eq. (1.11) becomes significant. when  $\chi^2 \sim 3H|\dot{\phi}(0)|/gm$ . But it may be that  $g^2\chi^2$  has become of order  $H^2$  first, causing  $\phi$  to oscillate about the origin. Including both possibilities we have<sup>#6</sup>

$$\chi_{\text{nl}}^2 \simeq \min \left\{ \frac{3H\dot{\phi}(0)}{gm}, \frac{H^2}{g^2} \right\}. \quad (2.26)$$

It follows from Eqs. (1.10) and (1.6) that the right hand side of Eq. (2.26) is much less than  $\chi_{\text{vev}}^2$ , making the right hand side of Eq. (1.12) insignificant as advertised.

## 2.4 Energy density and pressure of $\chi$

We have seen that the gradient of  $\chi$  is negligible compared with its time-derivative, and in our adopted gauge the gradient of  $\phi$  vanishes. Ignoring the gradient, the energy density and pressure are  $\rho = \rho_\phi + \rho_\chi$  and  $p = p_\phi + p_\chi$  where

$$\rho_\phi(t) = [V_0 + V(\phi)] + \frac{1}{2}\dot{\phi}^2, \quad (2.27)$$

$$p_\phi(t) = -[V_0 + V(\phi)] + \frac{1}{2}\dot{\phi}^2, \quad (2.28)$$

$$\rho_\chi \simeq -\frac{1}{2}|m^2(t)|\chi^2 + \frac{1}{2}\dot{\chi}^2 \quad (2.29)$$

$$\simeq 0 \quad (2.30)$$

$$p_\chi \simeq \frac{1}{2}|m^2(t)|\chi^2 + \frac{1}{2}\dot{\chi}^2 \quad (2.31)$$

$$\simeq |m^2(t)|\chi^2 \simeq \dot{\chi}^2. \quad (2.32)$$

In an unperturbed universe the energy continuity equation holds;

$$\dot{\rho}(t) = -3H(t)(\rho(t) + p(t)). \quad (2.33)$$

To the extent that spatial gradients are negligible it holds at each location. With a generic choice of the slicing, denoted by a subscript  $g$ , we have

$$\dot{\rho}_g(\mathbf{x}, t) \simeq -3\frac{da_g(\mathbf{x}, t)}{dt}(\rho_g(\mathbf{x}, t) + p_g(\mathbf{x}, t)), \quad (2.34)$$

where  $a_g(\mathbf{x}, t)$  is the locally defined scale factor. We are working in the gauge defined in Section 1.4, which means that  $t$  is proper time and we can choose  $a(\mathbf{x}, t) = a(t)$ , the unperturbed scale factor. We therefore have

$$\dot{\rho}(\mathbf{x}, t) \simeq -3H(t)(\rho(\mathbf{x}, t) + p(\mathbf{x}, t)). \quad (2.35)$$

---

<sup>#6</sup>With the potential  $V(\phi) = \frac{1}{2}m_\phi^2\phi^2$  the right hand side is  $\min\{m_\phi^2, H^2\} = m_\phi^2$  which means that only the first possibility exists. The existence of the second possibility for a more general potential was missed in [4].

We are dealing with the linear era, which means that the right hand sides of Eqs. (1.11) and (1.12) are negligible. With its right hand side negligible, Eq. (1.11) describes a free field which means that it satisfies the energy continuity equation by itself;

$$\dot{\rho}_\phi \simeq -3H(\rho_\phi + p_\phi) = -3H\dot{\phi}^2. \quad (2.36)$$

The same must therefore be true for  $\chi$ ;

$$\dot{\rho}_\chi(\mathbf{x}, t) \simeq -3H(\rho_\chi(\mathbf{x}, t) + p_\chi(\mathbf{x}, t)) = -3H\dot{\chi}^2(\mathbf{x}, t). \quad (2.37)$$

At each location we have<sup>#7</sup>

$$\ddot{\chi}(\mathbf{x}, t) + 3H\dot{\chi}(\mathbf{x}, t) + m^2(t)\chi(\mathbf{x}, t) \simeq 0. \quad (2.38)$$

Using this equation to differentiate Eq. (2.29) we find

$$\dot{\rho}_\chi(\mathbf{x}, t) \simeq -3H(\rho_\chi(\mathbf{x}, t) + p_\chi(\mathbf{x}, t)) - \frac{1}{2} \frac{d|m^2(t)|}{dt} \chi^2(\mathbf{x}, t). \quad (2.39)$$

The second term of the right hand side violates the energy continuity equation. This apparent inconsistency between the field equations and the energy continuity equation occurs because the effect of the interaction term  $g^2\phi^2\chi^2$  is dropped in Eq. (1.11) (ie. the right hand side is set to zero) but kept in Eq. (1.12).

If we demand approximate consistency between the field equations and the energy continuity equation, we need the second term of Eq. (2.39) to be much smaller than the first. That condition is equivalent to

$$\frac{d|m(t)|}{dt} \ll |m(t)|H, \quad (2.40)$$

which is stronger than the adiabaticity conditions (2.5) and (2.6) (with  $k \ll a|m(t)|$ ). But there is no need to impose this stronger condition, because the near cancellation of the two terms of  $\rho_\chi$  makes it unreasonable to expect the approximate evolution (2.38) of  $\chi$  to give even an approximate estimate of  $\rho_\chi$ . By contrast, the right hand side of the energy continuity equation has no cancellation so that it can be used to evaluate  $\rho_\chi$ . Invoking Eq. (2.16), we find

$$\rho_\chi(\mathbf{x}, t) \simeq -\frac{3}{2}H|m(t)|\chi^2(\mathbf{x}, t). \quad (2.41)$$

## 2.5 Justifying the neglect of $\bar{\chi}$

It has been essential for our discussion that the spatial average of  $\chi(\mathbf{x}, t)$  is negligible. That is the case in a sufficiently large volume, because  $\chi(\mathbf{x}, t)$  is constructed entirely

---

<sup>#7</sup>In the present context Eq. (2.38) can be replaced by Eq. (2.16) by virtue of the adiabaticity condition. But in Section 6 we drop the adiabaticity condition.

from the Fourier modes. But to make contact with cosmological observations we should consider a finite box, whose (coordinate) size  $L$  is not too many orders of magnitude bigger than the size of the presently observable universe [19]. Denote the average within the box by  $\bar{\chi}$  we have

$$\chi(\mathbf{x}) \simeq \bar{\chi} + \chi_{>}(\mathbf{x}), \quad (2.42)$$

where the Fourier modes of  $\chi_{>}$  satisfy  $kL > 1$  so that

$$\langle \chi_{>}^2 \rangle = \int_{L^{-1}}^{\infty} \frac{dk}{k} \mathcal{P}_{\chi}(k). \quad (2.43)$$

The average within the box comes from modes with  $kL \lesssim 1$ , and for a random location of the box the expectation value of  $\bar{\chi}^2$  is

$$\langle \bar{\chi}^2 \rangle \simeq \int_0^{L^{-1}} \frac{dk}{k} \mathcal{P}_{\chi}(k). \quad (2.44)$$

To justify the neglect of  $\bar{\chi}$  we need  $\langle \bar{\chi}^2 \rangle \ll \langle \chi_{>}^2 \rangle$ . In our scenario, where  $\mathcal{P}_{\chi}(k)$  peaks at a value  $k_*(t)$ , this is equivalent to  $Lk_*(t) \gg 1$ . That is satisfied because the scale  $k_*(t)$  is supposed to be much smaller than the observable universe. To have  $\langle \bar{\chi}^2 \rangle \gtrsim \langle \chi_{>}^2 \rangle$  we would presumably have to allow the transition from  $m^2(t) = -H^2$  to  $m^2(t) = H^2$  to take at least several  $e$ -folds so that it can generate a contribution to  $\chi$  that has a flat spectrum. Then, if the flat spectrum generated during the transition dominates, one would have

$$\langle \bar{\chi}^2 \rangle / \langle \chi_{>}^2 \rangle \simeq N_{\text{before}} / N_{\text{after}}, \quad (2.45)$$

where  $N_{\text{before}}$  ( $N_{\text{after}}$ ) is the number of  $e$ -folds of transition before (after) the observable universe leaves the horizon.

### 3. Contribution $\zeta_{\chi}$ to the curvature perturbation

We write the contribution to the curvature perturbation that is generated during the waterfall as  $\zeta_{\chi} = \zeta_{\text{lin}} + \zeta_{\text{nl}}$ , where the first term is generated during the linear era, and the second is generated afterward up to some epoch just after inflation has ended.

The curvature perturbation is  $\zeta(\mathbf{x}, t) \equiv \delta \ln a(\mathbf{x}, t)$ , where  $a(\mathbf{x}, t)$  is the locally defined scale factor on the spacetime slicing of uniform  $\rho$ . As in Eq. (2.35), the spatial gradient is supposed to be negligible, which in general requires smoothing on a super-horizon scale. Using that equation, we see that the change in  $\zeta$  between any two times  $t_1$  and  $t_2$  is

$$\zeta(\mathbf{x}, t_2) - \zeta(\mathbf{x}, t_1) = \delta N(\mathbf{x}, t_1, t_2), \quad (3.1)$$

where  $N(\mathbf{x}, t_1, t_2)$  where  $N$  is the number of  $e$ -folds between slices of uniform  $\rho$ . Working to first order in  $\delta\rho$ ,<sup>#8</sup> we will use this result to calculate  $\zeta_{\text{lin}}$ , and then see how it might be used to calculate  $\zeta_{\text{nl}}$ .

We note in passing that an equivalent procedure is to integrate the expression

$$\dot{\zeta}(\mathbf{x}, t) = -\frac{H(t)}{\rho(t) + p(t)} \delta p_{\text{nad}}(\mathbf{x}, t), \quad (3.2)$$

where

$$\delta p_{\text{nad}}(\mathbf{x}, t) \equiv \delta p(\mathbf{x}, t) - \frac{\dot{p}(t)}{\dot{\rho}(t)} \delta \rho(bfx, t). \quad (3.3)$$

### 3.1 The contribution $\zeta_{\text{lin}}$

During the linear era, the gradient of  $\rho$  is negligible without any smoothing. We are working in a gauge where  $\delta\phi = 0$  so that  $\rho(\mathbf{x}, t) = \rho_\chi(\mathbf{x}, t) + \rho_\phi(t)$ . Ignoring the inhomogeneity of the locally defined Hubble parameter,

$$\zeta_{\text{lin}}(\mathbf{x}, t) = H [\delta t(\mathbf{x}, t_{\text{end}}) - \delta t(\mathbf{x}, t_{\text{start}})], \quad (3.4)$$

and

$$H\delta t(\mathbf{x}, t) \equiv -H \frac{\delta \rho_\chi(\mathbf{x}, t)}{\dot{\rho}(t)} = \frac{1}{3} \frac{\delta \rho_\chi(\mathbf{x}, t)}{\langle \dot{\chi}^2(t) \rangle + \dot{\phi}^2(t)}. \quad (3.5)$$

Using Eq. (2.41),

$$H\delta t(\mathbf{x}, t) = \frac{1}{3} \frac{\rho_\chi(t)}{\langle \dot{\chi}^2(t) \rangle} \frac{\langle \dot{\chi}^2(t) \rangle}{\langle \dot{\chi}^2(t) \rangle + \dot{\phi}^2(t)} \frac{\delta \chi^2(\mathbf{x}, t)}{\langle \chi^2(t) \rangle} \quad (3.6)$$

$$= -\frac{1}{2} \frac{H}{|m(t)|} \frac{\langle \dot{\chi}^2(t) \rangle}{\langle \dot{\chi}^2(t) \rangle + \dot{\phi}^2(t)} \frac{\delta \chi^2(\mathbf{x}, t)}{\langle \chi^2(t) \rangle}. \quad (3.7)$$

Using Eq. (2.21), we have for  $k \ll k_*(t)$

$$H^2 \sqrt{\pi} \mathcal{P}_{\delta t}(k, t) = \left( \frac{H}{2|m(t)|} \right)^2 \left( \frac{\langle \dot{\chi}^2(t) \rangle}{\langle \dot{\chi}^2(t) \rangle + \dot{\phi}^2(t)} \right)^2 \left( \frac{k}{k_*(t)} \right)^3. \quad (3.8)$$

We assume that  $|\delta t(t_{\text{end}})| \gg |\delta t(t_{\text{start}})|$ , which will be justified within the approximation of Section 5. Then we have

$$\zeta_{\text{lin}}(\mathbf{x}) = -H\delta t(\mathbf{x}, t_{\text{end}}) \quad (3.9)$$

$$= -\frac{H}{2|m(t_{\text{end}})|} \frac{\langle \dot{\chi}^2(t_{\text{end}}) \rangle}{\langle \dot{\chi}^2(t_{\text{end}}) \rangle + \dot{\phi}^2(t_{\text{end}})} \frac{\delta \chi^2(\mathbf{x}, t_{\text{end}})}{\langle \chi^2(t_{\text{end}}) \rangle}. \quad (3.10)$$

---

<sup>#8</sup>A second-order calculation of  $\zeta$  is needed only to treat very small non-gaussianity corresponding to reduced bispectrum  $|f_{\text{NL}}| \lesssim 1$ . On cosmological scales, such non-gaussianity will eventually be measurable (and is expected if  $\zeta$  comes from a curvaton-type mechanism [5]). But there is no hope of detecting such non-gaussianity on much smaller scales.

The inhomogeneity of  $H$  is indeed negligible because it generates a contribution

$$\int_{t_{\text{start}}}^{t_{\text{end}}} \delta H(t, \mathbf{x}) dt = \frac{H}{2\rho_\chi} \int_{t_{\text{start}}}^{t_{\text{end}}} \delta \rho(\mathbf{x}, t) dt \simeq \frac{H \delta \rho_\chi(\mathbf{x}, t_{\text{end}})}{2\rho |m(t_{\text{end}})|}. \quad (3.11)$$

This is much less than  $H\delta t(t_{\text{end}})$  in magnitude, because  $|m(t)| \gg H$  and  $|\dot{\rho}| \ll H\rho$ .

Using Eq. (3.8), we have for  $k \ll k_*(t_{\text{end}})$

$$\sqrt{\pi} \mathcal{P}_{\zeta_{\text{lin}}}(k) = \left( \frac{H}{2|m(t_{\text{end}})|} \right)^2 \left( \frac{\langle \dot{\chi}^2(t_{\text{end}}) \rangle}{\langle \dot{\chi}^2(t_{\text{end}}) \rangle + \dot{\phi}^2(t_{\text{end}})} \right)^2 \left( \frac{k}{k_*(t_{\text{end}})} \right)^3 \ll \left( \frac{k}{k_*(t_{\text{end}})} \right)^3. \quad (3.12)$$

In the opposite regime  $k \gg k_*(t_{\text{end}})$ ,  $\zeta_{\text{lin}}$  is negligible because  $\delta\chi^2$  is. Therefore,  $\mathcal{P}_{\zeta_{\text{lin}}}$  peaks at  $k_*(t_{\text{end}})$  with the value

$$\sqrt{\pi} \mathcal{P}_{\zeta_{\text{lin}}}(k_*(t_{\text{end}})) \simeq \left( \frac{H}{2|m(t_{\text{end}})|} \right)^2 \left( \frac{\langle \dot{\chi}^2(t_{\text{end}}) \rangle}{\langle \dot{\chi}^2(t_{\text{end}}) \rangle + \dot{\phi}^2(t_{\text{end}})} \right)^2. \quad (3.13)$$

If  $m^2(t_{\text{end}}) \simeq -m^2$ , Eq. (2.25) holds and we have

$$\langle \dot{\chi}^2(t_{\text{end}}) \rangle / \dot{\phi}^2(t_{\text{end}}) \simeq 12M_{\text{P}}^2 H^2 / \dot{\phi}^2(t_{\text{end}}) \gg 1. \quad (3.14)$$

(The inequality follows from Eq. (1.6)). Then Eq. (3.12) simplifies to

$$\sqrt{\pi} \mathcal{P}_{\zeta_{\text{lin}}}(k) = \left( \frac{H}{2m} \right)^2 \left( \frac{k}{k_*(t_{\text{end}})} \right)^3. \quad (3.15)$$

### 3.2 The contribution $\zeta_{\text{nl}}$

Let us estimate the number of  $e$ -folds  $N_{\text{nl}}$  after the end of the linear era. At  $t_{\text{end}}$ ,  $\chi^2$  is increasing exponentially. Soon afterward it starts to affect  $\phi$ , driving it towards zero. We therefore expect  $m^2(t)$  to quickly approach  $-m^2$  after  $t_{\text{end}}$  (if it is not there already), restoring at least approximately the linear evolution of  $\chi^2$ . Then Eq. (2.16) will hold with  $|m(t)| \sim m$  giving

$$N_{\text{nl}} \sim (H/m) \ln(\chi_{\text{vev}}/\chi_{\text{end}}). \quad (3.16)$$

If the linear era ends only when the right hand side of Eq. (1.12) becomes important we have  $\ln(\chi_{\text{vev}}/\chi_{\text{end}}) \sim 1$ , giving  $N_{\text{nl}} \sim H/m \ll 1$ . But if it ends when the right hand side of Eq. (1.11) becomes important we may have  $\ln(\chi_{\text{vev}}/\chi_{\text{end}}) \gg 1$  which allows  $N_{\text{nl}} \gtrsim 1$ .

Now we consider the contribution  $\zeta_{\text{nl}}$ , that is generated between  $t_{\text{end}}$  and some time  $t_2$  just after inflation has everywhere ended. To calculate it we need to smooth on a super-horizon scale. Then we can use the  $\delta N$  formula which gives

$$\zeta_{\text{nl}}(\mathbf{x}, t) = H\delta t_{12}(\mathbf{x}), \quad (3.17)$$

where the initial and final slices both have uniform  $\rho$  and  $t_{12}(\mathbf{x})$  is the proper time interval between them.

At each location, the linear era ends at the epoch  $t_{\text{nl}}(\mathbf{x})$  given by Eq. (2.23). At this epoch there is nearly-exponential inflation, and inflation ends at some later time  $t_{\text{noninf}}(\mathbf{x})$ . If  $\Delta t(\mathbf{x}) \equiv t_{\text{noninf}}(\mathbf{x}) - t_{\text{nl}}(\mathbf{x})$  is sufficiently small it can be taken to correspond to a spacetime slice of negligible thickness. Then  $\delta t_{12}$  is given by the ‘end-of-inflation’ formula [20]

$$\delta t_{12}(\mathbf{x}) \simeq \frac{\delta \rho(\mathbf{x})}{\dot{\rho}(t_{\text{end}})}, \quad (3.18)$$

where  $\delta \rho(\mathbf{x})$  is defined on the slice. The addition of  $\zeta_{\text{nl}}$  to  $\zeta_{\text{lin}}$  corresponds to taking the final slice of the  $\delta N$  formula be the transition slice, instead of a slice of uniform  $\rho$ .

This equation is valid to first order in  $\delta \rho$ . To derive it we take the separation between the initial and final slice to be not much bigger than is needed for them to enclose the transition slice. Then we can take the unperturbed quantity  $\dot{\rho}(t)$  to have a constant value both during inflation and non-inflation. This gives

$$\delta t_{12}(\mathbf{x}) \simeq \delta \rho(\mathbf{x}) \left( \frac{1}{\dot{\rho}_{\text{inf}}} - \frac{1}{\dot{\rho}_{\text{noninf}}} \right). \quad (3.19)$$

Since  $|\dot{\rho}_{\text{inf}}|$  is evaluated during nearly-exponential inflation, it is much smaller than  $|\dot{\rho}_{\text{noninf}}|$  leading to Eq. (3.18).

We are defining  $\delta \rho_\chi$  on a slice of uniform  $\rho_\phi$  and  $\delta \rho$  is defined on a slice of uniform  $\chi$ . The time displacement from the first slice to the second slice is  $-\delta \rho_\chi / \dot{\rho}_\chi$ , which means that  $\delta \rho = \delta \rho_\chi \dot{\rho}_\phi(t_{\text{end}}) / \dot{\rho}_\chi(t_{\text{end}})$ . Putting this into Eq. (3.18) we get

$$\zeta_{\text{nl}}(\mathbf{x}) / \zeta_{\text{lin}}(\mathbf{x}) \simeq \dot{\rho}_\phi(t_{\text{end}}) / \dot{\rho}_\chi(t_{\text{end}}) \simeq \dot{\phi}^2(t_{\text{end}}) / \dot{\chi}^2(t_{\text{end}}). \quad (3.20)$$

We are only interested in the case that this ratio is  $\gtrsim 1$ . Then the inclusion of  $\zeta_{\text{nl}}$  corresponds to omitting the middle term of Eq. (3.10). From Eq. (3.14), this case can occur only if  $|m(t_{\text{end}})| \ll m$ .

Now comes a crucial point. From the derivation of Eq. (3.18), it is clear that the criterion for its validity is  $\Delta t(\mathbf{x}) \ll |\delta t_{12}(\mathbf{x})|$ , at a typical location. (In words, the thickness of the transition slice is negligible compared compared with its warping.) This simple remark has not been made before, and consequently it has not been checked whether the criterion is satisfied.

In our case,  $H \Delta t(\mathbf{x})$  is given by Eq. (3.16) with  $\chi_{\text{end}}^2$  replaced by  $\chi_{\text{nl}}^2(\mathbf{x})$ . Since that quantity appears only in the log the change will not have much effect, and we will have  $H \Delta t(\mathbf{x}) \sim N_{\text{nl}}$  at a typical location. On the other hand, the typical value of  $|\zeta_{\text{nl}}(\mathbf{x})| = H |\delta t_{12}(\mathbf{x})|$  is  $\mathcal{P}_{\zeta_{\text{nl}}}^{1/2}(k)$  where  $k \lesssim aH$  is the smoothing scale used to define



$\zeta_{\text{nl}}$ . The criterion for Eq. (3.18) to be valid is therefore  $N_{\text{nl}} \ll \mathcal{P}_{\zeta_{\text{nl}}}^{1/2}(k)$ . In the regime of interest  $\dot{\phi}^2(t_{\text{end}})/\dot{\chi}^2(t_{\text{end}}) \gg 1$ , this criterion becomes

$$\frac{|m(t_{\text{end}})|}{m} \ln \left( \frac{\chi_{\text{vev}}}{\chi_{\text{end}}} \right) \ll \left( \frac{k}{k_*(t_{\text{end}})} \right)^{3/2} \quad (3.21)$$

Whenever the criterion (3.21) is not satisfied, the calculation of  $\zeta_{\text{end}}$  that we have described does not apply.

### 3.3 Other uses of the ‘end of inflation’ formula

Our use of Eq. (3.18) to evaluate  $\zeta_{\text{nl}}$  is quite different from its usual applications [20, 21, 22, 23, 24]. In those applications, the field causing  $\delta\rho(\mathbf{x})$  has a nearly flat spectrum, leading to a nearly flat  $\mathcal{P}_{\zeta_{\text{nl}}}(k)$  that can give a significant (even dominant) contribution to  $\mathcal{P}_{\zeta}(k)$  on cosmological scales. Since  $\mathcal{P}_{\zeta}^{1/2}(k) \sim 5 \times 10^{-5}$  on these scales, Eq. (3.21) on cosmological scales becomes

$$N_{\text{tran}} \ll \mathcal{P}_{\zeta_{12}}^{1/2} < 5 \times 10^{-5}, \quad (3.22)$$

where  $N_{\text{tran}}$  now refers to the duration of the transition slice in the scenario under consideration, and  $\zeta_{12} = H\delta t_{12}$  is the contribution to  $\zeta$ .

Most of the other applications [20, 21, 22] consider hybrid inflation, with the transition slice the entire hybrid inflation waterfall. Of course their setup is different from ours because they introduce a third field, the one that generates  $\delta\rho$  in Eq. (3.18). In these cases,  $N_{\text{tran}}$  in Eq. (3.22) becomes the total duration of the waterfall. We see from Eq. (3.16) that it cannot be much less than  $H/m$ , which means that Eq. (3.22) needs  $H/m \ll 5 \times 10^{-5}$ . Since we need  $(H/m)^2 \gg H/M_{\text{P}}$  (corresponding to  $\lambda \ll 1$ ), this requires a fairly low inflation scale  $H \ll 10^{-9} M_{\text{P}}$ .

An alternative possibility [23] is for the transition slice to be at the end of thermal inflation [25, 26, 27, 28]. (Thermal inflation is a few  $e$ -folds of inflation occurring typically long after the usual inflation, which is ended by a thermal phase transition.) Then we expect roughly  $\Delta N \sim H/m$ , where  $m$  is the tachyonic mass of the field causing the end of thermal inflation. This criterion (3.22) is satisfied by the usual realizations of thermal inflation. Further possibilities for the transition slice are considered in [24].

## 4. Effect of $\zeta_{\chi}$

### 4.1 Cosmological black hole bound on $\mathcal{P}_{\zeta}$

The most dramatic effect of  $\zeta$  would be the formation of black holes. This places an upper bound on  $\mathcal{P}_{\zeta}$ , which we now discuss taking on board for the first time the non-gaussianity of  $\zeta$ .

The bound that we are going to consider rests on the validity of the following statement: if, at any epoch after inflation, there are roughly spherical and horizon-sized regions with  $\zeta$  significantly bigger than 1, a significant fraction of them will collapse to form roughly horizon-sized black holes.<sup>#9</sup> The validity is suggested by the following argument: the overdensity at horizon entry is  $\delta\rho/\rho \sim \zeta$ , and if it is of order 1 then  $\delta\rho \sim \rho = 3M_{\text{P}}^2 H^2$ . The excess energy within the Hubble distance  $H^{-1}$  is then  $M \sim H^{-3}\rho \sim M_{\text{P}}^2/H$ , which means that the Hubble distance corresponds roughly to the Schwarzschild radius of a black hole with mass  $M$ . The validity is confirmed by detailed calculation using several different approaches, as summarized for instance in [29].

Before continuing we mention the following caveat. Practically all of the literature, as well as the simple argument just given, assumes that  $\zeta$  within the region is not *very much* bigger than 1. Then the spatial geometry within the region is not too strongly distorted and the size of the black hole is indeed roughly that of the horizon. In the opposite case, the background geometry is strongly distorted and the wavenumber  $k$  defined in the background no longer specifies the physical size of the region at the epoch  $aH = k$  of horizon entry [30]. An entirely different discussion would then be necessary, which has not been given in the literature. As the opposite case does not arise in typical early-universe scenarios we ignore it.

We are interested in the case that  $\mathcal{P}_\zeta(k)$  has a peak at some value  $k_{\text{peak}}$ , and we assume that the width of the peak in  $\ln k$  is roughly of order 1 so that

$$\langle \zeta^2 \rangle = \int_0^\infty \mathcal{P}_\zeta(k) dk/k \simeq \mathcal{P}_\zeta(k_{\text{peak}}). \quad (4.1)$$

Regions with  $\zeta \gtrsim 1$  that might form black holes will be rare if  $\mathcal{P}_\zeta(k_{\text{peak}})$  is not too big. Observation demands that the regions must indeed be rare, because it places a strong upper bound on the fraction of space that can collapse to form horizon-sized black holes, on the assumption that the collapse takes place at a single epoch as is the case in our scenario. A recent investigation of the bound is given in [29], with extensive references to the literature. The bound depends on the epoch of collapse. Denoting it by  $\beta$  it lies in the range

$$10^{-20} \lesssim \beta \lesssim 10^{-5}. \quad (4.2)$$

To bound  $\mathcal{P}_\zeta(k_{\text{peak}})$ , we shall require  $y < \beta$  where  $y$  is the fraction of space with  $\zeta > \zeta_c$ , and  $\zeta_c$  is roughly of order 1.

The fraction  $y$  can be calculated from  $\langle \zeta^2 \rangle$  if we know the probability distribution of  $\zeta(\mathbf{x})$ . The standard assumption is that it is gaussian. Then

$$y = \frac{1}{2} \text{erfc}(\zeta_c / \sqrt{2\langle \zeta^2 \rangle}), \quad (4.3)$$

---

<sup>#9</sup>We are choosing the background scale factor  $a(t)$  so that the perturbation  $\zeta = \delta(\ln a(\mathbf{x}, t))$  has zero spatial average.

and using the large- $x$  approximation  $\text{erfc}(x) \simeq e^{-x^2}/\sqrt{\pi}x \sim e^{-x^2}$  we find

$$\mathcal{P}_\zeta(k_{\text{peak}}) \simeq \langle \zeta^2 \rangle \lesssim \zeta_c^2/2 \ln(1/f). \quad (4.4)$$

For the range (4.2) this gives (with  $\zeta_c \simeq 1$ )  $\mathcal{P}_\zeta(k_{\text{peak}}) \lesssim 0.01$  to  $0.04$ .

But  $\zeta_{\text{lin}}$  given by Eqs. (2.19) and (3.10) is actually non-gaussian, of the form

$$\zeta = -(g^2 - \langle g^2 \rangle). \quad (4.5)$$

With this form, there is no region of space where  $\zeta > \langle g^2 \rangle$ , and  $y \ll 1$  now implies some bound  $\langle g^2 \rangle - \zeta_c \ll \zeta_c$  which is practically equivalent to  $\langle g^2 \rangle < \zeta_c$ . This corresponds to  $\mathcal{P}_\zeta(k_{\text{peak}}) \lesssim 2$

$$\mathcal{P}_\zeta(k_{\text{peak}}) \simeq \langle \zeta^2 \rangle = 2\langle g^2 \rangle^2 \lesssim 2\zeta_c^2 \lesssim 2. \quad (4.6)$$

For completeness, we see what happens if  $\zeta = +(g^2 - \langle g^2 \rangle)$  with  $g$  gaussian. (This might be the case [31] if  $\zeta$  is generated after inflation by a curvaton-type mechanism.) Then we have

$$\mathcal{P}_\zeta(k_{\text{peak}}) \sim \langle \zeta^2 \rangle = 2\langle g^2 \rangle^2 \lesssim 2 \left[ \frac{\zeta_c}{2 \ln(1/y)} \right]^2, \quad (4.7)$$

which gives  $\mathcal{P}_\zeta(k_{\text{peak}}) \lesssim 2 \times 10^{-4}$  to  $2 \times 10^{-3}$ .

In all three cases, the bound on  $\mathcal{P}_\zeta(k_{\text{peak}})$  is very insensitive to  $f$  which means that it depends only weakly on the value of  $\beta$ . Turning that around though, the black hole abundance is very sensitive to  $\mathcal{P}_\zeta(k_{\text{peak}})$  which suggests that fine-tuning of parameters will be needed to get an eventually observable (yet presently allowed) abundance.

If the peak has width  $\Delta \ln k$  different from 1,  $\langle \zeta^2 \rangle \simeq \mathcal{P}_\zeta(k_{\text{peak}}) \Delta \ln k$ . If  $\Delta \ln k \ll 1$  this weakens the bound on  $\mathcal{P}_\zeta(k_{\text{peak}})$  by a factor  $(\Delta \ln k)^{-1}$ , but such a narrow peak is not generated in typical scenarios. If instead  $\Delta \ln k \gg 1$ , one might think that the bound on  $\mathcal{P}_\zeta(k_{\text{peak}})$  is strengthened by a factor  $(\Delta \ln k)^{-1}$ , but that conclusion is too hasty because the observational bound (4.2) refers to the formation of horizon sized black holes at a more or less definite epoch whereas the broad peak will lead to the formation of such black holes over  $\Delta \ln k$  Hubble times. The value of  $\langle \zeta^2 \rangle$  in that case is not directly related to the black hole abundance, and the black hole bound on  $\mathcal{P}_\zeta(k_{\text{peak}})$  is unlikely to be strengthened very much. For instance, if the observational bound on black hole abundance applies separately to the black holes formed within each unit interval of  $\ln k$ , the effective value of  $y$  for a given value of  $\mathcal{P}_\zeta(k_{\text{peak}})$  is just multiplied by that factor, which has a negligible effect on the bound on  $\mathcal{P}_\zeta(k_{\text{peak}})$ .

## 4.2 The effect of $\zeta_{\text{lin}}$

Now we discuss the effect of  $\zeta_{\text{lin}}$ , assuming that it is at least not canceled by  $\zeta_{\text{nl}}$ . By virtue of Eq. (2.5), the first term of Eq. (3.12) is  $\ll 1$ , and the second term is  $\leq 1$ .

If  $k_*(t_{\text{end}})$  is super-horizon,  $\zeta$  is of the form Eq. (4.5) with the minus sign, and the black hole bound is  $\mathcal{P}_\zeta \lesssim (k_*(t_{\text{end}})) \lesssim 2$ . This is likely to be well satisfied.

If instead  $k_*(t_{\text{end}})$  is sub-horizon, we have to remember that the black hole bound refers to horizon-sized regions. To apply it, we must drop sub-horizon modes of  $\zeta_{\text{lin}}$ . Estimating the bispectrum, trispectrum as in [4], one sees that this makes  $\zeta_{\text{lin}}$  nearly gaussian. Then  $\mathcal{P}_{\zeta_{\text{lin}}}$  peaks at  $k \sim k_{\text{end}} \equiv a(t_{\text{end}})H$ , and the black hole bound is roughly  $\mathcal{P}_{\zeta_{\text{lin}}}(k_{\text{end}}) \lesssim 10^{-2}$ . This too will be satisfied if  $k_*(t_{\text{end}})$  is well within the horizon.

We emphasize that these bounds refers to the formation of *horizon-sized* black holes. If  $k_*(t_{\text{end}})$  is sub-horizon, smaller black holes may also be formed. A discussion of their abundance would require assumptions about the evolution of the perturbations during the transition from inflation to non-inflation, and would be much more difficult than the corresponding discussion [32] for the formation of black holes from  $\zeta_\phi$ .

Although  $\mathcal{P}_{\zeta_{\text{lin}}}(k)$  is probably too small to form black holes, it may still be quite large. If reheating after inflation is long delayed this may lead to copious structure formation with a variety of possible cosmological effects [33].

Finally, let us see whether  $\mathcal{P}_{\zeta_{\text{lin}}}(k)$  can be significant on cosmological scales; ie. whether it can be comparable with the observed quantity  $\mathcal{P}_\zeta \simeq 10^{-9}$ . It follows from Eq. (2.15) that the scale  $k_*(t_{\text{end}})$  is shorter than the scale leaving the horizon at the beginning of the waterfall. Therefore, the inequality (3.12) implies that  $\mathcal{P}_{\zeta_{\text{lin}}}$  will give a negligible contribution to the observed quantity  $\mathcal{P}_\zeta \sim 10^{-9}$  if the shortest cosmological scale leaves the horizon more than  $3 \ln(10) \simeq 7$   $e$ -folds before the start of the waterfall, ie. if the observable universe leaves the horizon more than  $\simeq 22$   $e$ -folds before the start of the waterfall. We will see that this is assured within the approximation of Section 5.

## 5. Estimates using a simple approximation

In this section we make a simple approximation for  $m^2(t)$ . This will allow us to verify some of the assumptions that we have been making, especially if we assume that  $\phi$  satisfies the slow-roll approximation.

### 5.1 The approximation for $m^2(t)$

The approximation is

$$m^2(t) \simeq -\mu^3 t \quad (0 \lesssim \mu^3 t < m^2) \quad (5.1)$$

$$m^2(t) \simeq -m^2 \quad (\mu^3 t > m^2), \quad (5.2)$$

$$\mu^3 \equiv 2gm|\dot{\phi}(0)|. \quad (5.3)$$

The cross-over between the two expressions is at  $t = t_{\pm} \equiv m^2/\mu^3$ . The second expressions corresponds to setting  $\phi = 0$ . If the linear era ends at  $t < t_{\pm}$  only the first approximation is invoked.

The first approximation is exact at  $t = 0$ , and it ignores the time-dependence of  $d(\phi^2)/dt = 2\phi\dot{\phi}$ . The constancy of  $\phi$  is a good approximation at  $t \ll t_{\pm}$ , and so is the constancy of  $\dot{\phi}$  if (5.8) is sufficiently well satisfied. The fast transition requirement described in Section 1.3 is  $H \lesssim \mu$ . To simplify some of the estimates we will usually take the requirement to be

$$H \ll \mu \quad (\text{fast transition}). \quad (5.4)$$

With this approximation for  $|m^2(t)|$ , the linear era is completely described by the four parameters  $g$ ,  $H$ ,  $m$ , and  $\mu$ . Let us define  $N(t) \equiv Ht$ . Then the epoch  $t = t_{\pm}$  corresponds to

$$N(t_{\pm}) = \left(\frac{m}{\mu}\right)^2 \frac{H}{\mu} = \left(\frac{m}{\mu}\right)^3 \frac{H}{m} = \left(\frac{m}{H}\right)^2 \left(\frac{H}{\mu}\right)^3. \quad (5.5)$$

## 5.2 Slow-roll approximation

To obtain the strongest possible results, we assume that the evolution of  $\phi$  satisfies the slow-roll approximation, at least during some era that begins before the waterfall and ends when  $\phi$  ceases to affect the evolution of  $\chi$ .

Then unperturbed inflaton field satisfies

$$\ddot{\phi}(t) + 3H(t)\dot{\phi}(t) + V'(\phi(t)) = 0. \quad (5.6)$$

The basic slow-roll approximation is

$$3H\dot{\phi} \simeq -V'(\phi), \quad (5.7)$$

or equivalently

$$H|\ddot{\phi}/\dot{\phi}| \ll 1. \quad (5.8)$$

The requirement that the first derivative of Eq. (5.7) be consistent with Eq. (5.8) is  $\epsilon_H + \eta \ll 1$  where  $\epsilon_H \equiv |\dot{H}|/H^2$  and  $\eta \equiv V''/3H^2$ . The slow-roll approximation assumes  $\epsilon_H \ll 1$  and  $|\eta| \ll 1$ .

Before the waterfall, and for as long afterward as  $\chi$  has a negligible effect on  $\phi$ , the Fourier components of the perturbation  $\delta\phi$  satisfy

$$\ddot{\delta\phi}_{\mathbf{k}} + 3H(t)\delta\dot{\phi}_{\mathbf{k}} + (k/a)^2\delta\phi_{\mathbf{k}} + V''(\phi(t))\delta\phi_{\mathbf{k}} = 0. \quad (5.9)$$

This equation ignores back-reaction, which is a good approximation by virtue of the slow-roll approximation [5].

As a scale leaves the horizon, the vacuum fluctuation of  $\phi$  is converted to a classical perturbation  $\delta\phi$  with spectrum  $\simeq (H/2\pi)^2$ . At a given epoch, the vacuum fluctuation is set to zero on sub-horizon scales.

These results hold both before and during the waterfall. Focusing on the former era we have more results, because  $\phi$  is the only field. First, we have a couple more relations:

$$3M_{\text{P}}^2 H^2(t) = \rho_\phi \simeq V \equiv V_0 + V(\phi) \quad (5.10)$$

$$\epsilon_H = \frac{1}{2} \frac{\dot{\phi}^2}{M_{\text{P}}^2 H^2} \simeq \epsilon \equiv M_{\text{P}}^2 (V'/V)^2/2. \quad (5.11)$$

Second, we have the crucial result that  $\delta\phi$  generates nearly gaussian curvature perturbation  $\zeta_\phi$  to the curvature perturbation  $\zeta$ , with spectrum given by

$$\mathcal{P}_{\zeta_\phi}^{1/2}(k) \simeq \left. \frac{H^2}{2\pi\dot{\phi}} \right|_{aH=k}. \quad (5.12)$$

For a given  $k$  the spectrum is generated at the epoch of horizon exit  $k = aH$ , and is constant thereafter until at least the beginning of the waterfall.

### 5.3 Trading $\mu$ for $f$

For single-field inflation, we can use Eq. (5.12) to obtain more powerful results by trading  $\mu$  for [4]

$$f \equiv (5 \times 10^{-5})^{-1} H^2/2\pi\dot{\phi}(0) = (5 \times 10^{-5})^{-1} \mathcal{P}_{\zeta_\phi}^{1/2}(k_{\text{beg}}), \quad (5.13)$$

where  $k_{\text{beg}}$  is the horizon scale at the beginning of the waterfall. Inflation models are usually constructed so that  $\mathcal{P}_{\zeta_\phi}$  accounts for the observed  $\mathcal{P}_\zeta$  on cosmological scales. Then, if  $\mathcal{P}_{\zeta_\phi}$  is nearly scale-independent we will have  $f \sim 1$ . More generally there is an upper bound

$$f \lesssim 2 \times 10^3 \quad (\text{black hole constraint}) \quad (5.14)$$

corresponding to the black hole bound  $\mathcal{P}_{\zeta_\phi} \lesssim 10^{-2}$  on the spectrum of the nearly gaussian  $\zeta = \zeta_\phi$  that exists at the beginning of the waterfall. There is also a lower bound corresponding to Eq. (1.6):

$$f \gg 10^{-2} H/(10^{-5} M_{\text{P}}) \quad (\text{nearly exponential inflation}). \quad (5.15)$$

The relation between  $f$  and  $\mu$  is given by

$$\left( \frac{H}{\mu} \right)^3 \simeq 10^{-4} \frac{fH}{gm}.. \quad (5.16)$$

We are demanding  $g \ll 1$ , but the fast transition requirement  $H \ll \mu$  can always be satisfied because  $f < 2 \times 10^{-3}$  and  $H \ll m$ .

In this paper we are not specifying the potential  $V(\phi)$ . Most previous work considers the potential  $V(\phi) = \frac{1}{2}m_\phi^2\phi^2$ . Then slow-roll requires  $m_\phi \ll H$  and  $3H\dot{\phi} = -m_\phi^2\phi$ . The fast transition requirement  $H \lesssim \mu$  becomes

$$\left(\frac{m_\phi^2}{H^2}\right) \left(\frac{m^2}{H^2}\right) \gtrsim 1, \quad (5.17)$$

and  $f$  is given by

$$f = 10^4 g \left(\frac{m_\phi^2}{H^2}\right)^{-1} \left(\frac{m}{H}\right)^{-1}. \quad (5.18)$$

In this case we need  $f \ll 1$ , to avoid a positive spectral tilt for  $\mathcal{P}_\zeta$  which would conflict with observation. The requirement that  $V(\phi)$  does not support inflation (so that inflation ends with the end of the waterfall) is  $\phi_c \lesssim 10M_{\text{P}}$ , which is guaranteed by Eq. (1.10).

#### 5.4 The case $t_{\text{end}} < t_*$

In this case  $m^2(t) \sim -\mu^3 t$ , and Eq. (2.26) becomes

$$\chi_{\text{nl}}^2 \simeq \min \left\{ \frac{3H\mu^3}{2g^2m^2}, \frac{H^2}{g^2} \right\}. \quad (5.19)$$

Using Eq. (2.26) and  $\dot{\chi}^2 \simeq |m^2(t)|\chi^2$ ,

$$\frac{\langle \dot{\chi}^2(t_{\text{end}}) \rangle}{\dot{\phi}^2(t_{\text{end}})} \simeq \min \left\{ 6N(t_{\text{end}}), 4N(t_{\text{end}}) \frac{g^2 H m^2}{\mu^3} \right\}. \quad (5.20)$$

Growth begins when Eq. (2.6) is satisfied, corresponding to  $\mu t_1 \simeq 1$ , and  $\mu t_{\text{start}} = 2\mu t_1 \simeq 2$  and  $H t_{\text{start}} \ll 1$ .

The case  $N(t_{\text{end}}) \ll 1$  is considered in [4]. We then have

$$k_*^2(t_{\text{end}}) = a^2(t_{\text{start}})\mu^2/2(\mu t_{\text{end}})^{1/2}. \quad (5.21)$$

Using Eqs. (2.18) and (2.8) this gives<sup>#10</sup>  $\chi_{\text{nl}}^2 \simeq \mu^2 \exp[(4/3)(\mu t_{\text{end}})^{3/2}]$ , ie.

$$\mu t_{\text{end}} \simeq \left( \ln \frac{\chi_{\text{nl}}}{\mu} \right)^{2/3}. \quad (5.22)$$

Assuming instead  $N(t_{\text{end}}) \gtrsim 1$ ,

$$k_*^2(t_{\text{end}}) \simeq a^2(t_{\text{start}})\mu^{3/2}H^{1/2}. \quad (5.23)$$

Using Eqs. (2.18) and (2.8) we again arrive at Eq. (5.22)<sup>#11</sup>

---

<sup>#10</sup>In the prefactor of Eq. (2.18) we drop a numerical factor and a factor  $\tau^{-3/2}$ , because these are negligible compared with the exponent.

<sup>#11</sup>Factors  $\mu/H$  are ignored in the prefactor of Eq. (2.18) because the exponent is  $\gtrsim \mu/H$ .

Our implicit assumption that the growth era starts well before  $t_{\text{end}}$  corresponds to  $\mu t_{\text{end}} \gg 1$ . This is equivalent to  $\chi_{\text{nl}} \gg \mu$  or

$$\min \left[ \left( \frac{H}{mgf^{1/5}} \right)^{5/3}, \left( 10^{-4} \frac{fH}{g^4 m} \right)^{2/3} \right] \gg 1. \quad (5.24)$$

The requirement  $t_{\text{end}} < t_{=}$  corresponds to

$$\mu t_{\text{end}} < \left( \frac{m}{H} \right)^2 \left( \frac{H}{\mu} \right)^2 = \left[ 10^{-4} \left( \frac{m}{H} \right)^2 \frac{f}{g} \right]^{2/3}. \quad (5.25)$$

To get an upper bound on  $\mu t_{\text{end}}$  we use  $\mu \gtrsim H$  and  $\chi_{\text{nl}}^2 \ll \chi_{\text{vev}}^2 \ll M_{\text{P}}^2$ , to find  $\mu t_{\text{end}} \lesssim (\ln(M_{\text{P}}/H))^{2/3}$ . Using the range (1.9) this gives an upper bound on  $\mu t_{\text{end}}$  that is of order 5 to 20 with the lower end of the range far more likely. The upper bound corresponds to  $N(t_{\text{end}}) \lesssim (H/\mu)(\ln(M_{\text{P}}/H))^{2/3}$  giving

$$N(t_{\text{end}}) \ll [\ln(M_{\text{P}}/H)]^{2/3} \ll \ln(M_{\text{P}}/H). \quad (5.26)$$

Using  $\langle \dot{\chi}^2(t_{\text{start}}) \rangle \sim \mu^4$  and Eqs. (5.16), (5.14), and (5.24),<sup>#12</sup> we find  $\langle \dot{\chi}^2(t_{\text{start}}) \rangle \ll \dot{\phi}^2(t_{\text{start}})$ . This justifies the assumption  $|\delta t(\mathbf{x}, t_{\text{end}})| \gg |\delta t(\mathbf{x}, t_{\text{start}})|$  made after Eq. (3.7) because the initial strong growth of  $\langle \dot{\chi}^2(t) \rangle / \dot{\phi}^2(t)$  will outweigh the slower variation of the other factors.

### 5.5 The case $t_{\text{end}} > t_{=}$

In this case  $m^2(t_{\text{end}}) \simeq -m^2$ . As we discussed in Section 3.1,  $\chi_{\text{nl}}^2$  is given by Eq. (2.25), and  $\mathcal{P}_{\zeta_{\text{lin}}}$  by Eq. (3.15). Growth starts, at the latest, at  $t_{=} + m^{-1}$ .<sup>#13</sup>

Our our approximation makes  $d|m(t)|/dt$  discontinuous at  $t = t_{=}$  in violation of the adiabaticity condition (2.5). In reality  $|m(t)|$  will be smooth around  $t = t_{=}$ . To avoid specifying a definite form for  $|m(t)|$ , we confine ourselves to the case  $N(t_{\text{end}}) \gtrsim 1$ . Then

$$k_*^2(t_{\text{end}}) \simeq a^2(t_{\text{start}})mH. \quad (5.27)$$

Assume first that the growth era starts before  $t_{=}$ . Then Eqs. (2.18) and (2.8) give<sup>#14</sup>

$$\chi_{\text{vev}}^2 \simeq H^2 \exp \left( 2m(t_{\text{end}} - t_{=}) + \frac{4}{3}(\mu t_{=})^{3/2} \right). \quad (5.28)$$

We also have

$$mt_{=} = (\mu t_{=})^{2/3} = (m/\mu)^3 \gg 1, \quad (5.29)$$

<sup>#12</sup>Only the first case of Eq. (5.24) need be invoked for this purpose.

<sup>#13</sup>Since growth always begins,  $\chi$  always becomes classical. We have no need of a purely quantum treatment, which would require an entirely different approach. (In [4] we noted  $\chi$  may fail to become classical within the regime  $t \ll t_{=}$ . Contrary to what was stated there, that does not imply that  $\chi$  may fail to become classical at all.)

<sup>#14</sup>Since  $mt_{\text{end}} \gg m/H$ , and we ignore factors of  $m/H$  in the prefactor in Eq. (2.18).



where the inequality holds because we are assuming that growth starts before  $t_+$ . Using the first equality we get

$$\chi_{\text{vev}}^2 \simeq H^2 \exp \left( 2m \left( t_{\text{end}} - \frac{1}{3} t_+ \right) \right) \simeq H^2 \exp (2m t_{\text{end}}). \quad (5.30)$$

The final approximation is  $t_+ \ll 3t_{\text{end}}$ , which should be adequate because we are in the regime  $t_+ < t_{\text{end}}$ . In this approximation, the growth before  $t_+$  has a negligible effect. Using it we find

$$\chi_{\text{vev}}^2 \sim H^2 e^{2m t_{\text{end}}}, \quad (5.31)$$

leading to<sup>#15</sup>

$$N(t_{\text{end}}) \equiv \frac{H}{m} (m t_{\text{end}}) \simeq \frac{H}{m} \ln(M_{\text{P}}/H). \quad (5.32)$$

This gives again the bound (5.26).

Now suppose that growth does not start before  $t_+$ . Then the inequality in Eq. (5.29) is reversed leading to  $N_+ \ll 1$ . We therefore arrive again at Eq. (5.31) leading to Eq. (5.32). In this case  $H t_{\text{start}} = N_+ + H/m$  which is  $\ll 1$  as before. Also, from Eqs. (2.13) and (5.27), we have  $k_*(t_{\text{start}})/k_*(t_{\text{end}}) \simeq H/m$ . Using Eq. (3.8), this ensures that the typical value of  $|\delta t(\mathbf{x}, t_{\text{end}})/\delta t(\mathbf{x}, t_{\text{start}})|$  is  $\gtrsim m/H \gg 1$ .

## 5.6 Duration of the non-linear era

For case  $t_{\text{end}} > t_+$ ,  $\chi_{\text{end}}^2$  is the value of  $\chi^2$  at which the right hand side of Eq. (1.12) becomes important, corresponding to  $\ln(\chi_{\text{vev}}/\chi_{\text{end}}) \sim 1$ . For the opposite case, Eqs. (5.19) and (5.14) give

$$\frac{\chi_{\text{vev}}^2}{\chi_{\text{end}}^2} \ll \frac{M_{\text{P}}^2}{mH}. \quad (5.33)$$

This is much less than  $M_{\text{P}}^2/H^2$ , which means that Eq. (3.16) gives  $N_{\text{nl}} \ll \ln(M_{\text{P}}/H)$ . This is the same bound that we obtained for  $N(t_{\text{end}})$ . It therefore applies to the total number of  $e$ -folds of the waterfall,  $N_{\text{water}} \equiv N(t_{\text{end}}) + N_{\text{nl}}$ .

As seen in Section 4.2, we need the waterfall to begin more than 22  $e$ -folds after the observable universe leaves the horizon, if we are to be sure that  $\mathcal{P}_{\zeta_{\text{lin}}}(k)$  has a negligible effect on cosmological scales. Equivalently we need  $N_{\text{obs}} - 22 > N_{\text{wat}}$ . From Eq. (1.1) the left hand side of this inequality is bigger than  $47 - [\ln(M_{\text{P}}/H)]/2$  and we have seen that the right hand side is  $\ll \ln(M_{\text{P}}/H)$ . The inequality will therefore hold if  $47 \gg [\ln(M_{\text{P}}/H)]/2$  ie. if  $H/M_{\text{P}} \gg 10^{-41}$ . This is hardly stronger than the BBN bound (1.9), which means that  $\mathcal{P}_{\zeta_{\text{lin}}}(k)$  is almost certainly negligible on cosmological scales.

## 5.7 Two inflation models

To illustrate the power of our results we apply them to two inflation models based on supersymmetry.

---

<sup>#15</sup>We ignore a factor  $H/m$  within the log, which is permissible since  $H/m$  is also the prefactor.

### 5.7.1 Supersymmetric GUT hybrid inflation

Supersymmetric GUT hybrid inflation [12, 13] takes  $\chi$  to be a GUT Higgs field so that  $\chi_{\text{vev}} \simeq 10^{-2}M_{\text{P}}$  corresponding to  $(H/m)^2 \simeq 10^{-5}$ . This is not small enough for the ‘end of inflation’ formula to yield the entire waterfall contribution to  $\zeta$  (Eq. (3.22)). Supersymmetry gives  $g^2 = 2\lambda$  leading to  $g^2 = 10^9(H/M_{\text{P}})^2$ . This leaves for our discussion two independent parameters which we take as  $g$  and  $f$ .

The potential  $V(\phi)$  may depend on several parameters [13]. It typically steepens, and our discussion applies only if the parameters are such that steepening does not end slow-roll before  $t_{\text{end}}$ . Requiring the inflaton perturbation to generate  $\zeta$  on cosmological scales, the steepening implies  $10^{-1.5}g \lesssim f \lesssim 1$ , the lower bound coming from Eq. (5.15). Using Eq. (5.16) we have  $\mu/H \simeq 10^2(g/f)^{1/3}$ . The fast transition requirement (5.16) is certainly satisfied if  $g^2 \gg 10^{-12}$  (ie.  $H/M_{\text{P}} \gg 10^{-10}$  which usually holds [13]).

The parameter space allows  $t_{\text{end}} < t_{=}$  (with either of the possibilities in Eq. (2.26)) as well as  $t_{\text{end}} > t_{=}$ . Provided that  $H/\mu$  is well below 1, the duration of the waterfall is quite short, and the ‘end of inflation’ formula can give  $\zeta_{\text{nl}}$  in part of the parameter space ((3.21)).

### 5.7.2 Supernatural/running-mass inflation

Supernatural inflation [14] and running-mass [15] inflation take  $\chi_{\text{vev}}$  roughly of order  $M_{\text{P}}$  corresponding to  $m$  roughly of order  $H$ . This can be motivated by supposing that  $\chi$  is a string modulus, with gravity-mediated or anomaly-mediated supersymmetry breaking [34]. The former case,  $V_0^{1/4} \sim 10^{10}$  GeV or  $H \sim 10^{-15}M_{\text{P}}$  is usually invoked and the latter would give  $H \sim 10^{-13}M_{\text{P}}$  or so. This low inflation scale and  $m \sim H$  are distinguishing features of the paradigm.

The potential for supernatural inflation is  $V(\phi) = m_\phi^2 \phi^2/2$  which does not allow  $\zeta = \zeta_\phi$  on cosmological scales. Running-mass inflation takes  $V(\phi)$  to be the renormalization group improved potential allowing  $\zeta = \zeta_\phi$  on cosmological scales which is assumed. In a suitable regime of parameter space,  $\zeta_\phi(k)$  on small scales can be big enough to exceed the cosmological bound on black hole formation, providing a constraint on the parameter space; in other words we can have  $f \sim 10^3$ . This is another distinguishing feature of the paradigm.

Since  $m$  is roughly of order  $H$  our criterion  $m^2 \gg H^2$  cannot be very well satisfied and the analysis of the next Section is really more appropriate. To proceed we assume that  $m/H$  is a bit above 1, and take  $f \sim 1$ . Then the fast transition requirement  $H \ll \mu$  is satisfied for  $g^2 \gg 10^{-8}$  which is expected.

Since  $m$  is roughly of order  $H$  and we deal with the case  $t_{\text{end}} > t_{=}$ , corresponding to  $m^2(t_{\text{end}}) \simeq -m^2$ . This gives  $N(t_{\text{end}}) \sim \ln(M_{\text{P}}/H) \sim 33$ , and  $k_*(t_{\text{end}})$  is outside the horizon, with  $\mathcal{P}_{\zeta_{\text{lin}}}(k_*(t_{\text{end}})) \simeq (H/2m)^2$ . This is not far below 1, and the black hole bound might be violated.

The duration of the non-linear era is  $N_{\text{nl}} \sim 1$ . Since Eq. (3.21) is not satisfied, the contribution  $\zeta_{\text{nl}}$  is not given by the ‘end of inflation’ formula.

## 6. The case $m \sim H$

Now we consider the case that  $m/H$  is  $\lesssim 1$  but not extremely small. To arrive at estimates we assume that  $\mathcal{P}_\chi(k)$  in this regime continues to peak at some value  $k_*(t) \ll a(t)|m(t)|$ . Since  $|m(t)| \leq m \sim H$  this means that  $k_*(t)$  is always outside the horizon.

We assume that the gradient of  $\chi$  is negligible, checking the self-consistency of that assumption later. Then Eq. (2.38) holds. Considering either of the two independent solutions we define  $s(t)$  by

$$\dot{\chi}(\mathbf{x}, t) = Hs(t)\chi(\mathbf{x}, t), \quad (6.1)$$

giving

$$s^2(t) + 3s(t) - |m^2(t)|/H^2 = -\dot{s}/H. \quad (6.2)$$

We assume that the right hand side of Eq. (6.2) is negligible, checking later for self-consistency. Keeping only the growing mode this gives

$$s(t) = -\frac{3}{2} + \sqrt{\frac{9}{4} + \frac{|m^2(t)|}{H^2}} \quad (6.3)$$

$$\simeq \frac{1}{3} \frac{|m^2(t)|}{H^2}. \quad (6.4)$$

The gradient of  $\chi$  is indeed negligible compared with  $\dot{\chi}$ , which means that  $\rho_\chi$  and  $p_\chi$  are given by Eqs. (2.29) and (2.31). Also, the assumption that the right hand side of Eq. (6.3) is negligible is self-consistent if and only if Eq. (2.40) holds. This means that the field equation (2.38) is consistent with the energy continuity equation.

If the approximation (5.1) holds, Eq. (2.40) becomes  $Ht \gg 1$ . But the same approximation inserted into Eq. (6.1) gives at  $Ht \gg 1$

$$Ht \sim (H/\mu)^2 \ln [\chi^2(\mathbf{x}, t)/\chi^2(\mathbf{x}, H^{-1})]. \quad (6.5)$$

Since we haven’t calculated  $\chi(\mathbf{x}, t)$  from the vacuum fluctuation we don’t know the precise value of  $\chi^2(\mathbf{x}, H^{-1})$  but it presumably lies roughly between  $\mu$  and  $m \sim H$  since these are the relevant mass scales. As we saw earlier this would make the log at most of order  $10^2$  or so. Therefore, since we are imposing  $H \ll \mu$ , Eq. (6.5) is hardly compatible with  $Ht \gg 1$ . The conclusion is that Eq. (2.40) probably requires the regime (5.2),  $m^2(t) \simeq -m^2$ , which we assume from now on. That in turn implies  $\langle \dot{\chi}^2(t_{\text{end}}) \rangle \gg \dot{\phi}^2(t_{\text{end}})$ .

To calculate  $\zeta_{\text{lin}}$  we use Eq. (3.4), and assume  $|\delta t(t_{\text{end}})/\delta t(t_{\text{start}})| \gg 1$ . Using Eqs. (2.29), (6.3), and (6.1), we find

$$\rho_\chi/\dot{\chi}^2 = 3/2s(t) \simeq H^2/m^2, \quad (6.6)$$

to be compared with  $\rho_\chi/\dot{\chi}^2 = 3H/2|m(t)|$  in the case  $m^2 \gg H^2$ . We therefore have

$$\zeta_{\text{lin}}(\mathbf{x}) \simeq -\frac{H^2}{m^2} \frac{\delta\chi^2(\mathbf{x}, t_{\text{end}})}{\langle\chi^2(t_{\text{end}})\rangle}. \quad (6.7)$$

Since  $\mathcal{P}_\chi(k)$  peaks at  $k_*$ , we expect that the final equality of Eq. (2.18) will be roughly correct. Also we expect that  $\mathcal{P}_{\delta\chi^2}(k)$  will be given roughly by Eq. (2.22) at  $k \lesssim k_*$  and will fall off at bigger  $k$ . Then, using Eq. (6.4) we see that  $\mathcal{P}_{\zeta_{\text{lin}}}(k)$  peaks at  $k \sim k_*(t_{\text{end}})$  with a value

$$\mathcal{P}_{\zeta_{\text{lin}}}(k_*(t_{\text{end}})) \simeq \frac{H^2}{m^2}. \quad (6.8)$$

We conclude that the black hole bound is likely to be violated if  $m$  is significantly below  $H$ .

A crucial feature of our setup is the condition (2.40), which is necessary for consistency if the gradient of  $\chi$  is negligible and there is no cancellation between the two terms of  $\rho_\chi$ . We found that the solution of Eq. (2.1) then indeed makes the gradient of  $\chi$  negligible, with no cancellation. But the solution of Eq. (2.1) may also make the gradient negligible with no cancellation, in a part of parameter space where with the condition (2.40) violated. In such a regime, we would have to conclude that the linear approximation leading to Eq. (2.1) is invalid.

## 7. Comparison with other calculations

Nineteen other papers have considered the contribution of the waterfall to  $\chi$  [14, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53]. in the fast transition regime. Some of them also consider the issue of black hole formation [14, 35, 51, 53], concluding that the black hole constraint is satisfied for  $m \gg H$  but not for  $m \sim H$ . That is roughly our conclusion though we are less sure. In view of this, one may wonder whether the present paper and its companion [4] are needed. They are, for several reasons.

First, all of the previous papers take  $\phi$  to be canonically normalized, and nearly all of them go much further by assuming  $V(\phi) = m_\phi^2\phi^2/2$ . Second, none of the previous papers specifies all of the assumptions that are made, as we do here and in [4]. Third, none of them except [51, 53] considers the non-gaussian black hole bound as we have done in the present paper. Fourth, most of them present a calculation which is much more complicated than ours. Finally, all of the other papers except perhaps [51, 53] have errors.

The last point was considered in our earlier paper [4]. The problem for many of the papers [14, 35, 36, 37, 38, 39, 40, 41, 42] is that the waterfall is treated as two-field inflation, without imposing the requirement  $\langle \bar{\chi}^2 \rangle \gg \langle \chi^2 \rangle$  that would be needed to justify such a treatment.<sup>#16</sup> As we have seen, this is not the case. It is only within the slow transition regime, considered in [52, 54, 55, 56], that one can expect to find a regime of parameter space that allows the waterfall to be treated as two-field inflation.

We will not repeat the analysis of the problems of the other papers [44, 45, 46, 47, 48, 49, 50] appearing before [4], that was given in the latter paper. After [4] three papers appeared [40, 51, 53]. The paper [40] is a continuation of [45]. Its main focus is on the case  $\phi_c \gtrsim 10M_P$ , where inflation continues after the waterfall, but that does not affect the contribution to  $\zeta_\chi$  generated during the waterfall, and Eq. (6.15) of [40] reproduces the expression for  $\zeta_\chi$  given in Eq. A(10) of [45].

The papers [51, 53] consider the case  $m \lesssim H$ , and they calculate  $\chi_k$  by numerical integration with the potential  $V(\phi) = m_\phi^2 \phi^2/2$ .<sup>#17</sup> Then they evaluate  $\zeta_{\text{lin}}$  by integrating Eq. (3.2), finding that  $m < H$  is definitely forbidden by the black hole bound. The calculation assumes that the gradient of  $\chi$  can be ignored when evaluation  $\rho_\chi$  and  $p_\chi$  but they don't investigate the compatibility of the evolution equation Eq. (2.1) with the energy continuity equation. However, their results for the case  $m = H$  (with the other parameters fixed at particular values) shown in their Figure 2 is in excellent agreement with our Eq. (6.1), assuming  $|m(t)| = m$ . Their result for  $\mathcal{P}_{\zeta_{\text{lin}}}(k_*(t_{\text{end}}), t_{\text{end}})$  with the same parameter choice, shown in their Figure 4, is also in agreement with ours, assuming in addition  $\langle \dot{\chi}^2(t_{\text{end}}) \rangle \gg \dot{\phi}^2(t_{\text{end}})$ . It therefore seems that for at least this parameter choice, their assumption that the gradient of  $\chi$  is negligible is justified, and that moreover the consistency condition (2.40) is satisfied. But there is no reason to think that the same is true in the entire parameter space, considered in their Figure 5. Regarding the black hole bound, they note that  $\zeta_{\text{lin}}$  has the non-gaussian form (4.5). In [51] they use  $\langle \zeta_{\text{lin}}^2 \rangle \lesssim 1$  instead of our  $\mathcal{P}_{\zeta_{\text{lin}}} \lesssim 1$ . As the width of the peak in  $\mathcal{P}_{\zeta_{\text{lin}}}(k)$  is rather broad, this will somewhat overestimate the region of parameter space forbidden by the cosmological bound on  $\mathcal{P}_{\zeta_{\text{lin}}}$  as we noted in Section 4.1. In [53] a more sophisticated procedure is used to obtain the black hole bound, but they don't estimate the theoretical error and it is unclear to us whether it represents an improvement on our rough estimate  $\mathcal{P}_\zeta \lesssim 1$ .

## 8. Conclusion

We have considered the contribution  $\zeta_{\text{lin}}$  to  $\zeta$ , that is generated during the linear era of the waterfall within the Standard Scenario. We gave a rather complete calculation

<sup>#16</sup>The papers [35, 39, 14, 43] set  $\bar{\chi}^2 = \langle \chi^2 \rangle$  while the others regard it as a free parameter.

<sup>#17</sup>With  $m \lesssim H$ , the fast transition requirement (5.17) conflicts with the slow-roll requirement  $m_\phi \ll H$ , but the two are roughly compatible with the choice  $(m_\phi/H)^2 = 10^{-1}$  of [51, 53].

for the case that the waterfall mass  $m$  is much bigger than  $H$ , and arrived at estimates for  $m \sim H$ .

Taking on board our discussion of the non-gaussian black hole bound, we concluded that the black hole bound will be satisfied for  $m \gg H$ , but that it may well be violated for  $m \lesssim H$ . The latter case will be further investigated in a future publication [57], by numerically integrating Eq. (2.1).

A lot more will have to be done before we have a complete understanding of contribution to  $\zeta$  generated during the waterfall. A fundamental problem is to handle the ultra-violet cutoff, that is needed to obtain finite values for the fields and for the energy density and pressure. Our procedure of keeping only the classical field modes is approximate, and it violates at some level the energy continuity equation. This and related issues are discussed for instance in [58]. A precise procedure is advocated in [59], but its relation to our approximate procedure is unclear.

An understanding of the ultra-violet cutoff will allow one to decide on the minimum value of  $\dot{\phi}(0)$  that allows an initial linear era [17, 4]. With that in place one would hopefully verify that the value invoked in the present calculation is big enough. But it will still be unclear how to evaluate the contribution to  $\zeta$  that is generated during inflation after the linear era ends, when it is not given by the ‘end of inflation’ contribution. A numerical simulation, even with reasonable simplifications, might well require one to consider a patch of the universe that is too big to handle.

## 9. Acknowledgments

The author acknowledges support from the Lancaster-Manchester-Sheffield Consortium for Fundamental Physics under STFC grant ST/J00418/1, and from UNILHC23792, European Research and Training Network (RTN) grant.

## References

- [1] B. A. Ovrut and P. J. Steinhardt, “Inflationary Cosmology And The Mass Hierarchy In Locally Supersymmetric Theories,” *Phys. Rev. Lett.* **53** (1984) 732; K. Enqvist and D. V. Nanopoulos, “Primordial Two Component Inflation,” *Nucl. Phys. B* **252** (1985) 508.
- [2] A. D. Linde, “Axions in inflationary cosmology,” *Phys. Lett. B* **259** (1991) 38.
- [3] E. J. Copeland, A. R. Liddle, D. H. Lyth, E. D. Stewart and D. Wands, “False vacuum inflation with Einstein gravity,” *Phys. Rev. D* **49**, 6410 (1994).
- [4] D. H. Lyth, “Contribution of the hybrid inflation waterfall to the primordial curvature perturbation,” *JCAP* **1107**, 035 (2011).

- [5] D. H. Lyth and A. R. Liddle, *The primordial density perturbation*, Cambridge University Press, 2009;  
<http://astronomy.sussex.ac.uk/~andrewl/PDP/errata.pdf>;  
<http://astronomy.sussex.ac.uk/~andrewl/PDP/extensions.pdf>.
- [6] C. Armendariz-Picon, T. Damour and V. F. Mukhanov, “k - inflation,” Phys. Lett. B **458** (1999) 209; J. Garriga and V. F. Mukhanov, “Perturbations in k-inflation,” Phys. Lett. B **458** (1999) 219.
- [7] L. Alabidi and D. H. Lyth, “Inflation models and observation,” JCAP **0605** (2006) 016.
- [8] D. H. Lyth and E. D. Stewart, “More varieties of hybrid inflation,” Phys. Rev. D **54**, 7186 (1996).
- [9] E. D. Stewart, “Mutated hybrid inflation,” Phys. Lett. B **345**, 414 (1995);
- [10] G. Lazarides and C. Panagiotakopoulos, “Smooth hybrid inflation,” Phys. Rev. D **52** (1995) 559.
- [11] A. D. Linde, “Eternal extended inflation and graceful exit from old inflation without Jordan-Brans-Dicke,” Phys. Lett. B **249** (1990) 18; F. C. Adams and K. Freese, “Double field inflation,” Phys. Rev. D **43** (1991) 353.
- [12] G. R. Dvali, Q. Shafi and R. K. Schaefer, ‘Large scale structure and supersymmetric inflation without fine tuning,” Phys. Rev. Lett. **73**, 1886 (1994) [arXiv:hep-ph/9406319].
- [13] L. Covi, G. Mangano, A. Masiero and G. Miele, “Hybrid inflation from supersymmetric SU(5),” Phys. Lett. B **424** (1998) 253; R. Jeannerot, S. Khalil and G. Lazarides, “New shifted hybrid inflation,” JHEP **0207** (2002) 069; R. Jeannerot, S. Khalil, G. Lazarides and Q. Shafi, “Inflation and monopoles in supersymmetric SU(4)<sub>C</sub> x SU(2)<sub>(L)</sub> x SU(2)<sub>(R)</sub>,” JHEP **0010** (2000) 012; B. Kyae and Q. Shafi, “Inflation with realistic supersymmetric SO(10),” Phys. Rev. D **72** (2005) 063515; [arXiv:hep-ph/0504044]. G. Lazarides, I. N. R. Peddie and A. Vamvasakis, “Semi-shifted hybrid inflation with B-L cosmic strings,” Phys. Rev. D **78** (2008) 043518; M. U. Rehman, Q. Shafi and J. R. Wickman, “Supersymmetric Hybrid Inflation Redux,” Phys. Lett. B **683** (2010) 191; S. Khalil, M. U. Rehman, Q. Shafi and E. A. Zaakouk, “Inflation in Supersymmetric SU(5),” Phys. Rev. D **83** (2011) 063522;
- [14] L. Randall, M. Soljagic and A. H. Guth, “Supernatural inflation: Inflation from supersymmetry with no (very) small parameters,” Nucl. Phys. B **472** (1996) 377 [hep-ph/9512439].
- [15] E. D. Stewart, “Flattening the inflaton’s potential with quantum corrections,” Phys. Lett. B **391** (1997) 34; E. D. Stewart, “Flattening the inflaton’s potential

- with quantum corrections. II,” *Phys. Rev. D* **56** (1997) 2019; L. Covi, D. H. Lyth and A. Melchiorri, “New constraints on the running-mass inflation model,” *Phys. Rev. D* **67** (2003) 043507; L. Covi, D. H. Lyth, A. Melchiorri and C. J. Odman, “The running-mass inflation model and WMAP,” *Phys. Rev. D* **70** (2004) 123521.
- [16] D. H. Lyth, “Constraints on TeV-scale hybrid inflation and comments on non-hybrid alternatives,” *Phys. Lett. B* **466** (1999) 85.
- [17] J. F. Dufaux, G. N. Felder, L. Kofman and O. Navros, “Gravity Waves from Tachyonic Preheating after Hybrid Inflation,” *JCAP* **0903** (2009) 001.
- [18] D. H. Lyth, “Axions And Inflation: Sitting In The Vacuum,” *Phys. Rev. D* **45** (1992) 3394.
- [19] D. H. Lyth, “The curvature perturbation in a box,” *JCAP* **0712** (2007) 016.
- [20] F. Bernardeau and J. P. Uzan, “Inflationary models inducing non-gaussian metric fluctuations,” *Phys. Rev. D* **67** (2003) 121301; F. Bernardeau, L. Kofman and J. P. Uzan, “Modulated fluctuations from hybrid inflation,” *Phys. Rev. D* **70** (2004) 083004; D. H. Lyth, “Generating the curvature perturbation at the end of inflation,” *JCAP* **0511**, 006 (2005).
- [21] L. Alabidi and D. Lyth, “Curvature perturbation from symmetry breaking the end of inflation,” *JCAP* **0608** (2006) 006; D. H. Lyth and A. Riotto, “Generating the Curvature Perturbation at the End of Inflation in String Theory,” *Phys. Rev. Lett.* **97** (2006) 121301; M. P. Salem, “On the generation of density perturbations at the end of inflation,” *Phys. Rev. D* **72** (2005) 123516; F. Vernizzi and D. Wands, “Non-gaussianities in two-field inflation,” *JCAP* **0605** (2006) 019; D. H. Lyth and A. Riotto, “Generating the Curvature Perturbation at the End of Inflation in String Theory,” *Phys. Rev. Lett.* **97** (2006) 121301; L. Leblond and S. Shandera, “Cosmology of the Tachyon in Brane Inflation,” *JCAP* **0701** (2007) 009; B. Dutta, L. Leblond and J. Kumar, “Tachyon Mediated Non-Gaussianity,” *Phys. Rev. D* **78** (2008) 083522; M. Sasaki, “Multi-brid inflation and non-Gaussianity,” *Prog. Theor. Phys.* **120**, 159 (2008); A. Naruko and M. Sasaki, “Large non-Gaussianity from multi-brid inflation,” *Prog. Theor. Phys.* **121** (2009) 193; H. -Y. Chen, J. -O. Gong and G. Shiu, “Systematics of multi-field effects at the end of warped brane inflation,” *JHEP* **0809** (2008) 011; C. T. Byrnes, K. -Y. Choi and L. M. H. Hall, “Large non-Gaussianity from two-component hybrid inflation,” *JCAP* **0902** (2009) 017; C. -M. Lin, “Large non-Gaussianity generated at the end of Extended D-term Hybrid Inflation,” arXiv:0908.4168 [hep-ph]; E. Dimastrogiovanni, N. Bartolo, S. Matarrese and A. Riotto, “Non-Gaussianity and Statistical Anisotropy from Vector Field Populated Inflationary Models,” *Adv. Astron.* **2010** (2010) 752670; L. Alabidi, K. Malik, C. T. Byrnes and K. -Y. Choi, “How the curvaton scenario, modulated reheating and an inhomogeneous end of inflation are related,” *JCAP* **1011** (2010) 037;



- [22] S. Yokoyama and J. Soda, “Primordial statistical anisotropy generated at the end of inflation,” JCAP **0808** (2008) 005; K. Dimopoulos, M. Karčiauskas, D. H. Lyth and Y. Rodriguez, “Statistical anisotropy of the curvature perturbation from vector field perturbations,” JCAP **0905** (2009) 013; M. Karčiauskas, K. Dimopoulos and D. H. Lyth, “Anisotropic non-Gaussianity from vector field perturbations,” Phys. Rev. D **80** (2009) 023509; R. Emami and H. Firouzjahi, “Issues on Generating Primordial Anisotropies at the End of Inflation,”; JCAP **1201**, 022 (2012)
- [23] M. Kawasaki, T. Takahashi and S. Yokoyama, “Density Fluctuations in Thermal Inflation and Non-Gaussianity,” JCAP **0912** (2009) 012; T. Matsuda, “Cosmological perturbations from an inhomogeneous phase transition,” Class. Quant. Grav. **26**, 145011 (2009);
- [24] T. Matsuda, “Elliptic Inflation: Generating the curvature perturbation without slow-roll,” JCAP **0609** (2006) 003; T. Matsuda, “Brane inflation without slow-roll,” JHEP **0703** (2007) 096; T. Matsuda, “Generating curvature perturbations with MSSM flat directions,” JCAP **0706** (2007) 029;
- [25] G. Lazarides, C. Panagiotakopoulos and Q. Shafi, “Baryogenesis and the gravitino problem in superstring models,” Phys. Rev. Lett. **56** (1986) 557; D. H. Lyth and E. D. Stewart, “Cosmology with a TeV mass GUT Higgs,” Phys. Rev. Lett. **75** (1995) 201; D. H. Lyth and E. D. Stewart, “Thermal inflation and the moduli problem,” Phys. Rev. D **53** (1996) 1784; T. Barreiro, E. J. Copeland, D. H. Lyth and T. Prokopec, “Some aspects of thermal inflation: The Finite temperature potential and topological defects,” Phys. Rev. D **54** (1996) 1379;
- [26] K. Choi, E. J. Chun and J. E. Kim, “Cosmological implications of radiatively generated axion scale,” Phys. Lett. B **403** (1997) 209; E. J. Chun, D. Comelli and D. H. Lyth, “The Abundance of relativistic axions in a flaton model of Peccei-Quinn symmetry,” Phys. Rev. D **62** (2000) 095013; E. J. Chun, H. B. Kim and D. H. Lyth, “Cosmological constraints on a Peccei-Quinn flatino as the lightest supersymmetric particle,” Phys. Rev. D **62** (2000) 125001; E. J. Chun, H. B. Kim, K. Kohri and D. H. Lyth, “Flaxino dark matter and stau decay,” JHEP **0803** (2008) 061;
- [27] T. Asaka, M. Kawasaki and T. Yanagida, “Superheavy dark matter and thermal inflation,” Phys. Rev. D **60** (1999) 103518; T. Asaka and M. Kawasaki, “Cosmological moduli problem and thermal inflation models,” Phys. Rev. D **60** (1999) 123509; D. h. Jeong, K. Kadota, W. I. Park and E. D. Stewart, “Modular cosmology, thermal inflation, baryogenesis and predictions for particle accelerators,” JHEP **0411** (2004) 046; M. Kawasaki and K. Nakayama, “Late-time Affleck-Dine baryogenesis after thermal inflation,” Phys. Rev. D **74** (2006) 123508; G. N. Felder, H. Kim, W. I. Park and E. D. Stewart, “Preheating and Affleck-Dine leptogenesis after thermal inflation,” JCAP **0706** (2007) 005; R. Easther, J. T. . Giblin, E. A. Lim, W. I. Park and E. D. Stewart, “Thermal Inflation and the Gravitational Wave Background,” JCAP **0805** (2008) 013; S. Kim, W. I. Park and

- E. D. Stewart, “Thermal inflation, baryogenesis and axions,” *JHEP* **0901** (2009) 015; K. Choi, K. S. Jeong, W. I. Park and C. S. Shin, “Thermal inflation and baryogenesis in heavy gravitino scenario,” *JCAP* **0911** (2009) 018. W. I. Park, “A simple model for particle physics and cosmology,” *JHEP* **1007** (2010) 085; K. S. Jeong and M. Yamaguchi, “Axion model in gauge-mediated supersymmetry breaking and a solution to the  $\mu/B\mu$  problem,” *JHEP* **1107** (2011) 124; T. Moroi and K. Nakayama, “Domain Walls and Gravitational Waves after Thermal Inflation,” *Phys. Lett. B* **703** (2011) 160; R. Jinno, T. Moroi and K. Nakayama, “Imprints of Cosmic Phase Transition in Inflationary Gravitational Waves,” arXiv:1112.0084 [hep-ph].
- [28] G. R. Dvali, “Infrared hierarchy, thermal brane inflation and superstrings as superheavy dark matter,” *Phys. Lett. B* **459** (1999) 489; T. Matsuda, “Affleck-Dine baryogenesis after thermal brane inflation,” *Phys. Rev. D* **65** (2002) 103501; K. Dimopoulos and D. H. Lyth, “Models of inflation liberated by the curvaton hypothesis,” *Phys. Rev. D* **69** (2004) 123509; T. Matsuda, “Thermal hybrid inflation in brane world,” *Phys. Rev. D* **68** (2003) 047702; J. O. Gong, “Modular thermal inflation without slow-roll approximation,” *Phys. Lett. B* **637** (2006) 149; D. E. Morrissey and J. D. Wells, “Holomorphic selection rules, the origin of the  $\mu$  term, and thermal inflation,” *JHEP* **0701** (2007) 102; D. Spolyar, “SuperCool Inflation: A Graceful Exit from Eternal Inflation at LHC Scales and Below,” arXiv:1111.3629 [astro-ph.CO].
- [29] B. J. Carr, K. Kohri, Y. Sendouda and J. Yokoyama, “New cosmological constraints on primordial black holes,” *Phys. Rev. D* **81** (2010) 104019.
- [30] M. Kopp, S. Hofmann and J. Weller, “Separate Universes Do Not Constrain Primordial Black Hole Formation,” arXiv:1012.4369 [astro-ph.CO].
- [31] K. Kohri, D. H. Lyth and A. Melchiorri, “Black hole formation and slow-roll inflation,” *JCAP* **0804**, 038 (2008).
- [32] D. H. Lyth, K. A. Malik, M. Sasaki and I. Zaballa, “Forming sub-horizon black holes at the end of inflation,” *JCAP* **0601** (2006) 011; I. Zaballa and M. Sasaki, “Boosted perturbations at the end of inflation,” *JCAP* **1003** (2010) 002.
- [33] R. Easther, R. Flauger and J. B. Gilmore, “Delayed Reheating and the Breakdown of Coherent Oscillations,” *JCAP* **1104** (2011) 027, and earlier work cited there.
- [34] M. Dine, G. Festuccia, J. Kehayias and W. Wu, “Axions in the Landscape and String Theory,” *JHEP* **1101** (2011) 012.
- [35] J. Garcia-Bellido, A. D. Linde and D. Wands, “Density perturbations and black hole formation in hybrid inflation,” *Phys. Rev. D* **54**, 6040 (1996).
- [36] R. H. Brandenberger, A. R. Frey and L. C. Lorenz, “Entropy Fluctuations in Brane Inflation Models,” *Int. J. Mod. Phys. A* **24** (2009) 4327.

- [37] R. H. Brandenberger, K. Dasgupta and A. C. Davis, “A Study of Structure Formation and Reheating in the D3/D7 Brane Inflation Phys. Rev. D **78** (2008) 083502.
- [38] D. Mulryne, D. Seery and D. Wesley, “Non-Gaussianity constrains hybrid inflation,” arXiv:0911.3550 [astro-ph.CO].
- [39] A. A. Abolhasani and H. Firouzjahi, “No Large Scale Curvature Perturbations during Waterfall of Hybrid Phys. Rev. D **83** (2011) 063513.
- [40] A. A. Abolhasani, H. Firouzjahi and M. Sasaki, “Curvature perturbation and waterfall dynamics in hybrid inflation,” arXiv:1106.6315 [astro-ph.CO].
- [41] D. Parkinson, S. Tsujikawa, B. A. Bassett and L. Amendola, “Testing for double inflation with WMAP,” Phys. Rev. D **71** (2005) 063524;
- [42] S. Tsujikawa, D. Parkinson and B. A. Bassett, “Correlation - consistency cartography of the double inflation Phys. Rev. D **67** (2003) 083516;
- [43] A. A. Abolhasani, H. Firouzjahi and M. H. Namjoo, “Curvature Perturbations and non-Gaussianities from Waterfall Phase Class. Quant. Grav. **28** (2011) 075009.
- [44] J. Fonseca, M. Sasaki and D. Wands, “Large-scale Perturbations from the Waterfall Field in Hybrid Inflation,” JCAP **1009** (2010) 012.
- [45] J. O. Gong and M. Sasaki, “Waterfall field in hybrid inflation and curvature perturbation,” JCAP **1103** (2011) 028.
- [46] K. Enqvist and A. Vaihkonen, “Non-Gaussian perturbations in hybrid inflation,” JCAP **0409** (2004) 006.
- [47] N. Barnaby and J. M. Cline, “Nongaussian and nonscale-invariant perturbations from tachyonic preheating in hybrid inflation,” Phys. Rev. D **73** (2006) 106012.
- [48] N. Barnaby and J. M. Cline, “Nongaussianity from Tachyonic Preheating in Hybrid Inflation,” Phys. Rev. D **75** (2007) 086004.
- [49] J. M. Cline and L. Hoi, “Inflationary potential reconstruction for a WMAP running power spectrum,” JCAP **0606** (2006) 007.
- [50] K. Enqvist, A. Jokinen, A. Mazumdar, T. Multamaki and A. Vaihkonen, “Non-gaussianity from instant and tachyonic preheating,” JCAP **0503** (2005) 010;  
K. Enqvist, A. Jokinen, A. Mazumdar, T. Multamaki and A. Vaihkonen, “Cosmological constraints on string scale and coupling arising from JHEP **0508** (2005) 084.
- [51] E. Bugaev and P. Klimai, “Curvature perturbation spectra from waterfall transition, black hole constraints and non-Gaussianity,” arXiv:1107.3754 [astro-ph.CO].

- [52] T. Tanaka, T. Suyama and S. Yokoyama, “Use of delta N formalism - Difficulties in generating large local-type non-Gaussianity during inflation -,” *Class. Quant. Grav.* **27** (2010) 124003.
- [53] E. Bugaev and P. Klimai, “Formation of primordial black holes from non-Gaussian perturbations produced in waterfall transition,” arXiv:1112.5601 [astro-ph.CO].
- [54] S. Clesse, “Hybrid inflation along waterfall trajectories,” *Phys. Rev. D* **83** (2011) 063518.
- [55] H. Kodama, K. Kohri and K. Nakayama, “On the waterfall behavior in hybrid inflation,” arXiv:1102.5612 [astro-ph.CO].
- [56] D. Mulryne, S. Orani and A. Rajantie, “Non-Gaussianity from the hybrid potential,” arXiv:1107.4739 [hep-th].
- [57] A. A. Abolhasani, H. Firouzjahi, D. H. Lyth, M. Sasaki and D. Wands, in preparation.
- [58] L. Hollenstein, M. Jaccard, M. Maggiore and E. Mitsou, “Zero-point quantum fluctuations in cosmology,” arXiv:1111.5575 [astro-ph.CO].
- [59] J. Baacke, L. Covi and N. Kevlishvili, “Coupled scalar fields in a flat FRW universe: Renormalisation,” *JCAP* **1008** (2010) 026.